

Obrada signala u vremensko-frekvencijskoj domeni



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Vremensko-frekvencijske obrade

- Pod pojmom *spektar signala* redovito se podrazumijeva Fourierov spektar.
- Spektar snage slučajnih procesa također je definiran korištenjem Fourierove transformacije.
- Fourierov spektar je mjera izrazito prilagođena signalima nepromjenjivih svojstava, odnosno stacionarnim slučajnim procesima.

Fourierova transformacija

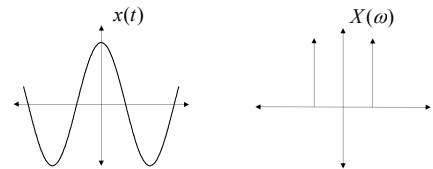
- Definicija:

$$\int_{-\infty}^{+\infty} \underbrace{x(t)}_{\text{signal}} \underbrace{e^{-j\omega t}}_{\substack{\text{kompleksna} \\ \text{harmonijska} \\ \text{funkcija}}} dt = X(\omega)$$

- $X(\omega)$ možemo promatrati kao mjeru sličnosti između $x(t)$ i $e^{-j\omega t}$.
- Ako je analizirana funkcija $x(t)$ harmonijska funkcija, sličnost će biti velika.

Fourierova transformacija

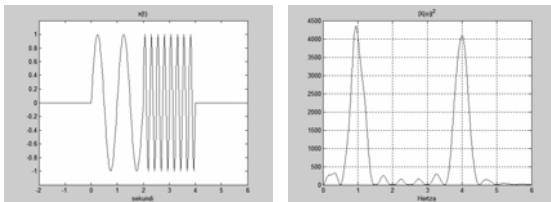
- Transformacija harmonijskog signala:



- Idealna lokalizacija u frekvencijskoj domeni.

Primjer promjenjivog signala

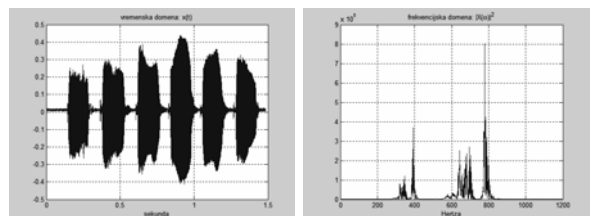
- Konstruirajmo signal promjenjivih svojstava u vremenskoj i frekvencijskoj domeni: dva sinusa.



- Vide se ili vremenski ili frekvencijski odnosi – niti jedan prikaz nije idealan.

Primjer stvarnog signala

- Akustički signal prikazan u dvije domene.
- Teško možemo prepoznati o čemu je riječ.



Diskretna Fourierova transformacija

- Diskretizacija vremena i/ili frekvencije:

$$t = mT \qquad \omega = n\Omega$$

$$X(\omega) = \sum_{m=-\infty}^{+\infty} x[mT] e^{-j\omega mT} \qquad X[n\Omega] = \int_{-\infty}^{+\infty} x(t) e^{-jn\Omega t} dt$$

FT diskretnih signala Fourierov red

$$t = mT, \quad \omega = n\Omega$$

$$X[n\Omega] = \sum_{m=-\infty}^{+\infty} x[mT] e^{-jn\Omega mT}$$

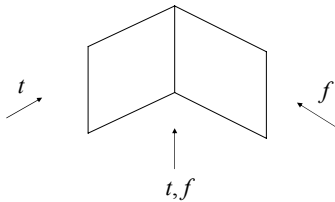
Diskretna Fourierova transformacija

- U svim slučajevima: linearna podjela (kvantizacija) vremenske i/ili frekvencijske osi.

Glazbena ljestvica

- Logaritamska podjela frekvencijske osi:
 - oktava – raspon od f_c do $2f_c$
 - ljestvica od 12 polutonova unutar jedne oktave
 - c, cis, d, dis, e, f, fis, g, gis, a, ais, h
$$f_{cis} = \sqrt[12]{2} f_c, \quad f_d = \sqrt[12]{2} f_{cis}, \quad \dots$$
- Raspon 220 – 440 Hz je isto što i 440 – 880 Hz ili 7040 – 14080 Hz: jedna glazbena oktava.
- Ljudsko uho bolje razlučuje NF od VF.
- Slično važi i za doživljaj slike.

Vremensko-frekvencijski prikaz



- Htjeli bismo konstruirati takvu transformaciju da istovremeno opažamo vremenske i frekvencijske odnose, s linearnim ili logaritamskim osima.

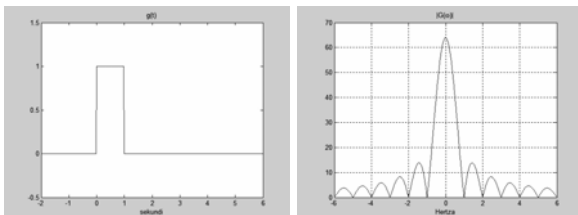
FT na vremenskom otvoru - STFT

- Konstruirajmo transformaciju s dobrim svojstvima lokalizacije u obadvojbe domene.
- Umjesto $e^{j\omega t} \rightarrow g(t-\tau) e^{j\omega t}$
- $g(t)$ – lokalni analizirajući otvor željenih svojstava u obje domene, τ – pomak.
- Rezultat: STFT.

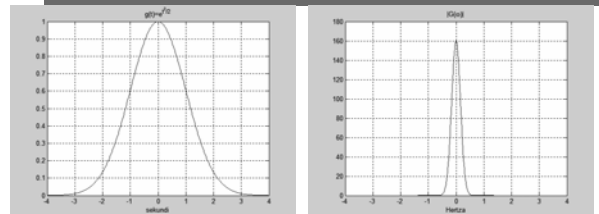
$$X(\tau, \omega) = \int_{-\infty}^{+\infty} x(t) g(t-\tau) e^{-j\omega t} dt$$

Primjeri vremenskih otvora

- Pravokutni otvor: dobra lokalizacija u vremenskoj, loša u frekvencijskoj domeni.



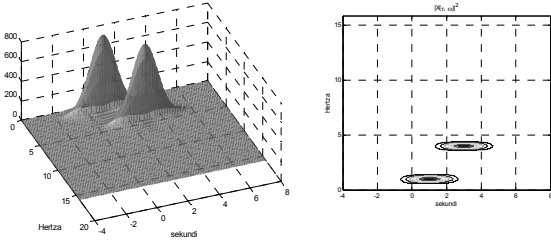
Gaussov otvor: ista funkcija u obje domene



- Širi vremenski \Leftrightarrow uži frekvencijski otvor.
- Heisenbergove relacije neodređenosti: produkt širina otvora \geq konst, a jednakost (tj. najmanji iznos) je upravo za Gaussov otvor.

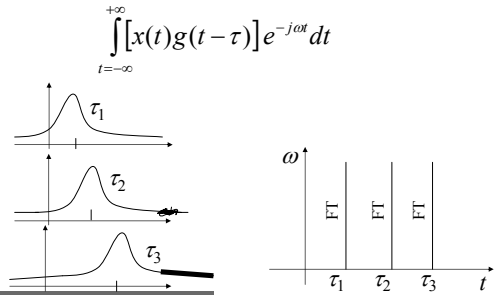
FT na vremenskom otvoru - STFT

- Rezultat STFT analize u T-F ravnini je ploha.
- Analizirani signal dva sinusa, Gaussov otvor:



Dvije interpretacije STFT-a

A) Signal lokaliziran u vremenu otvorom $g(t)$.

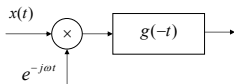


Dvije interpretacije STFT-a

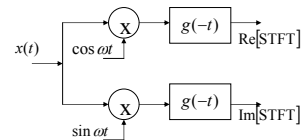
B) Signal pomaknut u frekvenciji i filtriran s $G^*(\omega)$.

$$\int_{t=-\infty}^{+\infty} \underbrace{[x(t) e^{-j\omega t}]}_{\substack{X(f+\omega) - \text{pomak} \\ \text{za } \omega \text{ u } f \text{ domeni}}} g[-(\tau-t)] dt$$

konvolucija



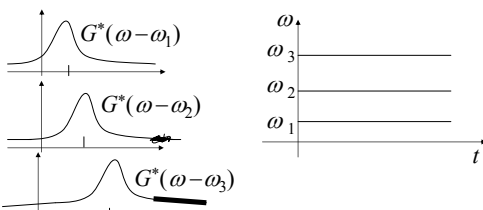
Dvije interpretacije STFT-a



- Realna varijanta sustava B) s prethodnog slidea.

Dvije interpretacije STFT-a

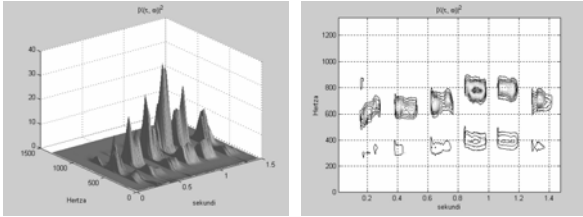
- Alternativna realizacija sustava B): umjesto frekventijskog pomicanja signala, uzmemo slog pomaknutih pojasno-propusnih filtara.



Spektrogram

- Koristimo linearni (ekvidistantni) filterski slog – jer je i širina analizirajućih $G^*(\omega - \omega_n)$ konstantna.
- $|X(\tau, \omega)|$ ili $|X(\tau, \omega)|^2$ naziva se spektrogramom.
- Prvi pokušaji analize govora elektromehaničkim spektrometrom iz 1930.g (slog BP filtra, lampice, fotosjetljivi papir).

STFT spektrogram stvarnog signala



- Analizirani fenomen lokaliziran u obje domene.
- Rezultat ovisi i o signalu i o analizirajućem otvoru.
- Otvor ima konstantna svojstva na T-F ravnini, zadanu rezoluciju – kompromis između vremena i frekvencije.

STFT spektar snage slučajnog procesa

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

- Spektar snage SP je mjera usrednjena u vremenu.
- $X_T(t)$ zamijenimo produktom $X(t)g(t-\tau)$ i izračunamo spektrogram kao funkciju τ i ω : $|X(\tau, \omega)|^2$.
- STFT spektar snage slučajnog procesa dobivamo primjenom operatora očekivanja (a bez vremenskog usrednjavanja):

$$S_{XX}(\tau, \omega) = E[|X(\tau, \omega)|^2].$$

- Rezultat: STFT spektar prati svojstva nestacionarnog procesa.

Problem inverzije STFT

- Fourierov transformacijski par:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} dt$$

- Postojanje inverzne formule jamči da nije došlo do gubitka informacije.
- Fourierova transformacija nije redundantna (ortogonalna je) i čuva energiju signala (uniformna).
- Postoji li inverzna STFT? Za koje vremenske otvore?

Inverzna STFT

- STFT jednodimenzionalnog signala je ploha $X(\tau, \omega)$: razlaganje je redundantno!

$$x(t) = \frac{1}{\|g\|^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(\tau, \omega) g(t-\tau) e^{j\omega t} d\omega d\tau$$

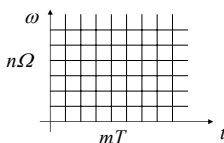
- Inverzija je moguća, ako je norma odabranog otvora $\|g\|$ različita od nule.
- Svaki $g(t)$ konačne energije je dobar!
- Poželjan izbor $\|g\|=1$, uz dobra lokalizacijska svojstva u vremenu i frekvenciji.

Diskretizacija STFT

- Umjesto na čitavoj plohi, STFT računamo na diskretnom skupu točaka.

$$g_{m,n}(t) = g\left(t - \underbrace{mT}_{\text{diskretni vremenski pomak}}\right) e^{jn\Omega t} \quad X[m, n] = \int_{-\infty}^{+\infty} x(t) g_{m,n}(t) dt$$

diskretni frekvencijski pomak



pravokutna kvantizacija T-F ravnine (otvor konstantnih svojstava)

Diskretizacija STFT

- Da li je razlaganje bez gubitka informacije, odnosno da li se iz diskretnog skupa koeficijenata $X[m, n]$ može restaurirati analizirani signal $x(t)$?

$$x(t) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} X[m, n] \cdot g_{m,n}(t)$$

- Rekonstrukcijska (inverzna) formula poznata je pod nazivom Gaborova ekspanzija signala.
- $X[m, n]$ je mjera sadržaja $x(t)$ na lokaciji $mT, n\Omega$.
- Kada je moguća rekonstrukcija?

Nužan uvjet rekonstrukcije

- $T\Omega > 2\pi$ podotipkavanje
 - rekonstrukcija nije moguća.
- $T\Omega = 2\pi$ granični (kritični) slučaj
 - moguća rekonstrukcija,
 - ne mogu se postići dobra svojstva u obje domene.
- $T\Omega < 2\pi$ nadotipkavanje
 - moguća rekonstrukcija,
 - redundantno razlaganje,
 - mogu se postići dobra svojstva u obje domene.

Dovoljan uvjet rekonstrukcije

- Rekonstrukcija je moguća i to na numerički stabilan način ako je $T\Omega \leq 2\pi$ i ako postoje dvije konstante A i B za koje vrijedi:

$$A \underbrace{\|x\|^2}_{\text{energija signala}} \leq \underbrace{\sum_{m,n} \|X[m,n]\|^2}_{\text{energija koeficijenata}} \leq B \underbrace{\|x\|^2}_{\text{energija signala}}$$

$$0 < A \leq B < \infty$$

- Razlaganje može biti i redundantno ili neuniformno, a konstante A i B zadaju energetski okvir transformacije (*engl. frame*).

Wavelet transformacija

- STFT – postiže lokalizaciju u obadvije domene, ali i dalje ima linearnu skalu.
- Wavelet funkcija $\psi^{a,\tau}(t) = \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-\tau}{a}\right)$ (“valić”)
 - $\Psi(t)$ – mother wavelet (prototip wavelet funkcije)
 - τ – vremenski linearni pomak
 - a – skala

$$\frac{1}{\sqrt{|a|}} \text{ – zbog normizacije } (\|\psi^{a,\tau}\| = \|\psi\|)$$

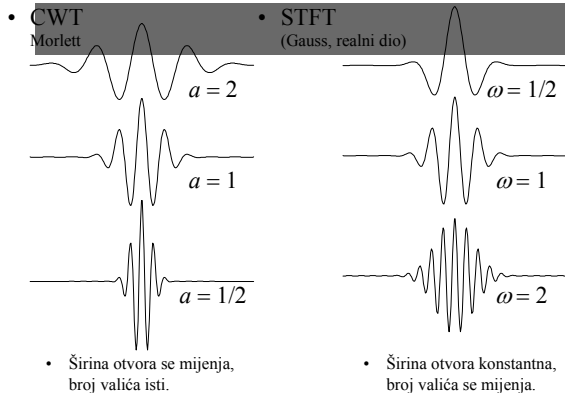
Wavelet transformacija

- Kontinuirana wavelet transformacija (CWT):

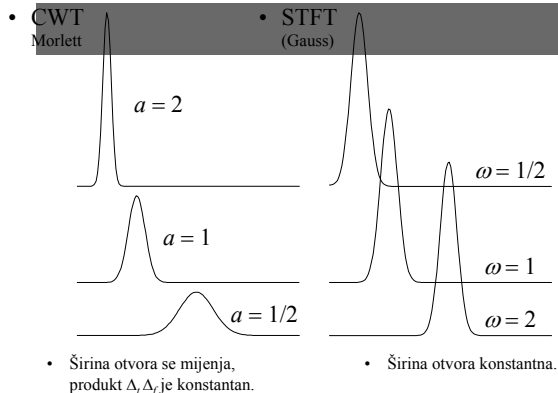
$$X(\tau, a) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} x(t) \psi\left(\frac{t-\tau}{a}\right) dt$$

- Funkcija razlaganja nije ograničena samo na kompleksnu harmonijsku funkciju $e^{j\alpha t}$.
- “Valić” $\psi(t)$ osigurava željena svojstva razlaganja.
- Analizirajuću funkciju pomičemo za τ , stežemo ili rastežemo za skalu a i uspoređujemo s $x(t)$.

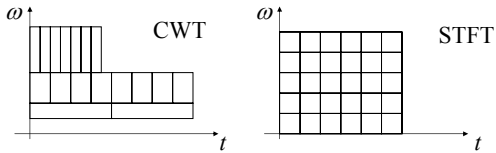
Usporedba CWT-STFT u T domeni



Usporedba CWT-STFT u F domeni



Rezolucije u T-F ravnini

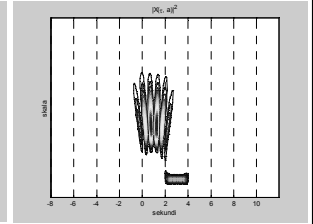
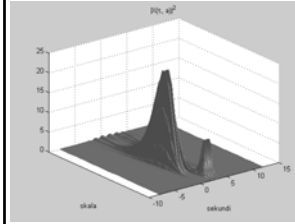


- STFT: konstantna rezolucija na cijeloj T-F ravnini
- CWT:
 - povećana rezolucija u frekencijskoj domeni za NF,
 - povećana rezolucija u vremenskoj domeni za VF.

CWT na primjeru

- Analizirani signal: dva sinusa, Morletov wavelet.

$$\psi(t) = e^{-\frac{t^2}{2}} \cos(5t)$$

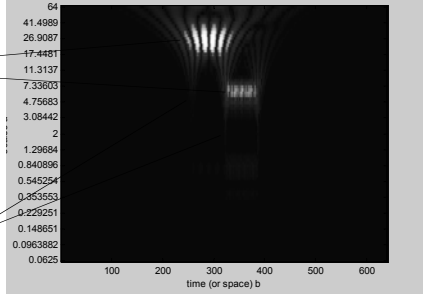


CWT na primjeru

Absolute Values of Ca,b Coefficients for a = 0.0625 0.068157 0.074325 0.081052 0.088388 ...

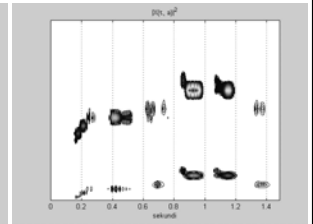
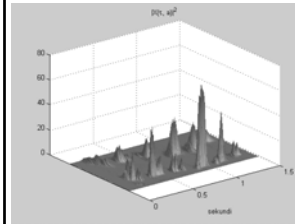
lokalizacija u frekvenciji
bolja za NF

Vrlo precizna lokalizacija u vremenu



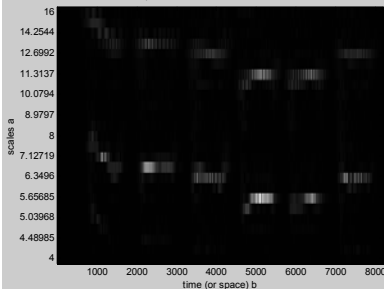
CWT na realnom primjeru

- Akustički signal, Morletov wavelet



CWT na realnom primjeru: gotovo “notni zapis”

Absolute Values of Ca,b Coefficients for a = 4 4.2379 4.4898 4.7568 5.0397 ...



Skalogram slučajnog procesa

- Mjeru $|\chi(\tau, a)|^2$ nazivamo skalogram.
- Skalogram slučajnog procesa dobivamo primjenom operatora očekivanja:

$$S_{xx}(\tau, a) = E[|\chi(\tau, a)|^2]$$

- Rezultat: skalogram prati svojstva nestacionarnog procesa, uz dobra svojstva Wavelet transformacije.

Inverzna CWT

$$x(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{a^2} X(\tau, a) \psi^{a,\tau}(t) da d\tau$$

$$C_\psi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|\psi(\omega)|^2}{\omega} d\omega$$

uvjet $C_\psi < \infty$ – uvjet prihvatljivosti

- Inverzija postoji ako je zadovoljen uvjet prihvatljivosti (*engl. admissibility condition*).
- Nužan uvjet $\Psi(0) = 0$: $\psi(t)$ ne smije sadržavati istosmjernu komponentu.

Neki zanimljivi $\psi(t)$

- Najbliži STFT-u je kompleksni Mallatov wavelet:

$$\psi(t) = e^{-\frac{t}{\xi_0}} e^{-j\pi t^2}$$

- Precizniji izraz (bez istosmjerne komponente) je:

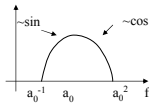
$$\psi(t) = \frac{1}{\sqrt[4]{\pi}} \left(e^{-\frac{t}{\xi_0}} - e^{-\frac{2t}{\xi_0}} \right) e^{-\frac{t^2}{\xi_0}}$$

- U praksi za $\xi_0 = 5$ ili veći drugi član u zagradi je zanemariv (tj. kad ima dovoljno valića).

Još neki zanimljivi $\psi(t)$

$$\psi(t) = (1-t^2)e^{-\frac{t^2}{2}}$$

- Mexican hat: derivacija Gaussa, bez istosmjerne vrijednosti.
- Meyerov wavelet – idealna lokalizacija u frekvencijskoj domeni.



Diskretna WT

$$a = a_0^m$$

- logaritamska podjela u skali (frekvenciji)

$$\tau = nT_0 a$$

- pomak usklađen s iznosom skale

$$tj. \tau = nT_0 a_0^m$$

Sve zajedno
$$\psi_{m,n}(t) = \frac{1}{\sqrt{a_0^m}} \psi\left(\frac{t}{a_0^m} - nT_0\right)$$

$$DWT \quad X[m,n] = \frac{1}{\sqrt{a_0^m}} \int_{-\infty}^{+\infty} x(t) \psi\left(\frac{t}{a_0^m} - nT_0\right) dt$$

Uvjet rekonstrukcije

$$x(t) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} X[m,n] \cdot \psi_{m,n}(t)$$

- Rekonstrukcija je moguća i to na numerički stabilan način ako postoje dvije konstante A i B za koje vrijedi:

$$A \underbrace{\|x\|^2}_{\text{energija signala}} \leq \sum_{m,n} \underbrace{\|X[m,n]\|^2}_{\text{energija koeficijenata}} \leq B \underbrace{\|x\|^2}_{\text{energija signala}}$$

$$0 < A \leq B < \infty$$

- Kod DWT-a ne postoji ekvivalent nužnog uvjeta $T\Omega \leq 2\pi$, koji je vrijedio za Gaborovu ekspanziju.

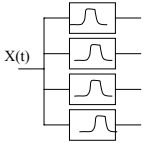
Oktavna DWT

- Kod waveleta se lako mogu pronaći ortogonalne baze s dobrim lokalizirajućim svojstvima u obje domene (što nije bio slučaj s Gaborom).
- Za realizaciju čest izbor je $a_0 = 2$ tj. oktavna podjela frekvencijske skale.
- Prednost: mogućnost brze realizacije filterskim slogovima.

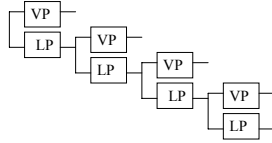
$$\psi_{m,n}(t) = \frac{1}{\sqrt{2^m}} \psi\left(\frac{t}{2^m} - nT_0\right)$$

Realizacija filtarskim slogovima

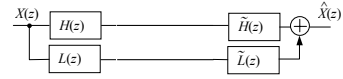
Kod STFT-a lako smo dobili ekvivalenciju s ekvidistantnim filtrom:



Kod waveleta ($a_0=2$) imat ćemo binarno stablo:



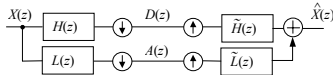
Filtarski slog s dva pojasa



- Potpuna rekonstrukcija: $\hat{X}(z) = z^{-\Lambda} X(z)$
- Uvjeti:

$$\tilde{L}(z)L(z) + \tilde{H}(z)H(z) = z^{-\Lambda}$$

Filtarski slog s dva pojasa



- Potpuna rekonstrukcija: $\hat{X}(z) = z^{-\Lambda} X(z)$
- Uvjeti:

$$\tilde{L}(z)L(z) + \tilde{H}(z)H(z) = 2z^{-\Lambda}$$

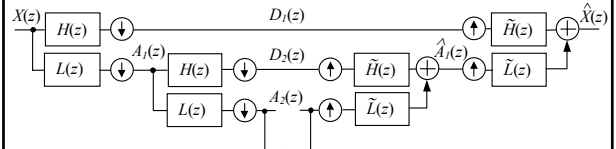
$$\tilde{L}(z)L(-z) + \tilde{H}(z)H(-z) = 0$$

- Biortogonalna rješenja (Vetterli 86)
 - različiti filtri na strani razlaganja i rekonstrukcije:

• a, b, c	→	$p, -q, r, -s, t$	$\tilde{H}(z) = L(-z)$
• p, q, r, s, t	↔	$-a, b, -c$	$\tilde{L}(z) = -H(-z)$

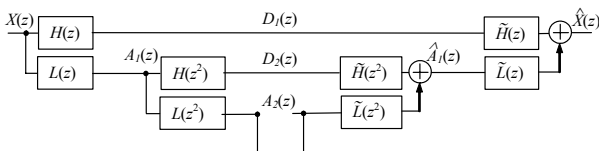
Realizacija DWT wavelet filtarskim slogom

- Wavelet stablo
- Realizira DWT diskretnih signala.



- Kakvi $H(z)$ i $L(z)$ zadovoljavaju, a da se radi o DWT filtarskom slogu (osim PR uvjeta)?

Nedecimirano wavelet stablo



- Umjesto decimacije imamo (ekvivalentno) potenciranje kompleksne varijable z .

$$\tilde{H}_N(z) = \tilde{H}(z^{2^{N-1}}) \prod_{i=1}^{N-2} \tilde{L}(z^{2^i})$$

$$\tilde{L}_N(z) = \prod_{i=1}^{N-1} \tilde{L}(z^{2^i})$$

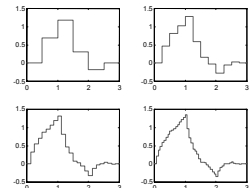
Wavelet funkcija i funkcija skale

- Konstruiramo pridružene funkcije, konstantne po odsječcima:

$$\tilde{\varphi}_N(t) = 2^{N/2} \tilde{l}_N[n], \quad \frac{n}{2^N} \leq t < \frac{n+1}{2^N}$$

$$\tilde{\psi}_N(t) = 2^{N/2} \tilde{h}_N[n], \quad \frac{n}{2^N} \leq t < \frac{n+1}{2^N}$$

- Ilustracija: Daubechies wavelet db2 za $n=1,2,3,4$.



Konvergencija pridruženih funkcija

- Ako pridružene funkcije konvergiraju i ako su limes funkcije zadovoljavajuće (glatke, ...), onda wavelet stablo realizira DWT diskretnih signala.

- Nužan uvjet konvergencije:

$$\tilde{L}(z)|_{z=-1} = 0, \quad \frac{1}{\sqrt{2}}\tilde{L}(z)|_{z=1} = 1$$

- Jedna varijanta dovoljnog uvjeta (Mallat 1989.):

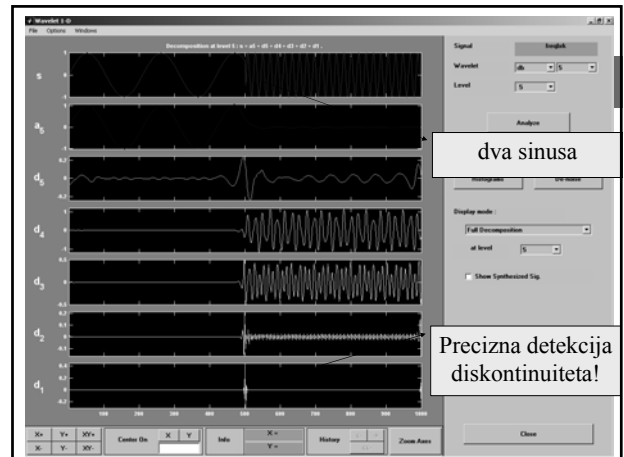
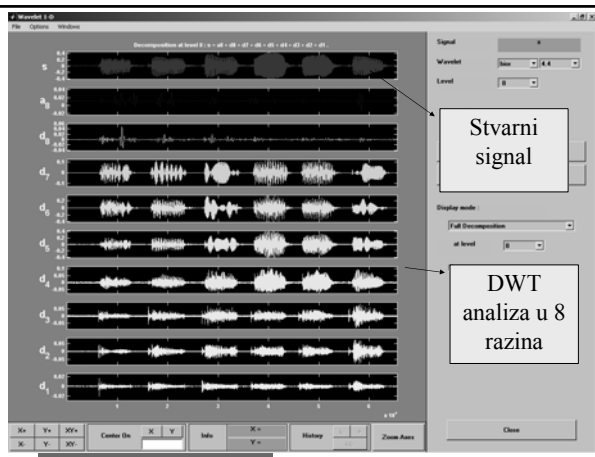
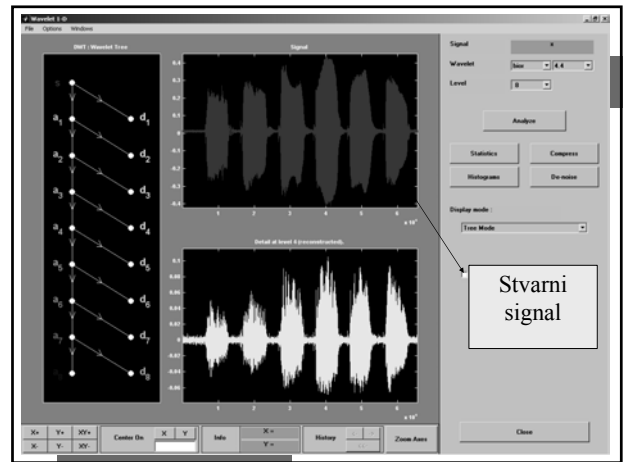
$$|\tilde{L}(\omega)| > 0, \quad \omega < \frac{\pi}{2}$$

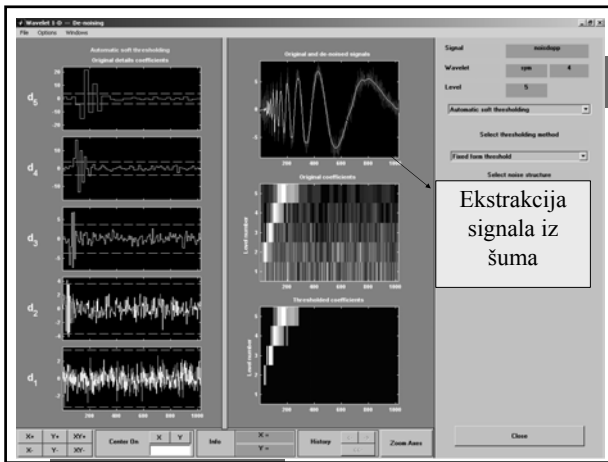
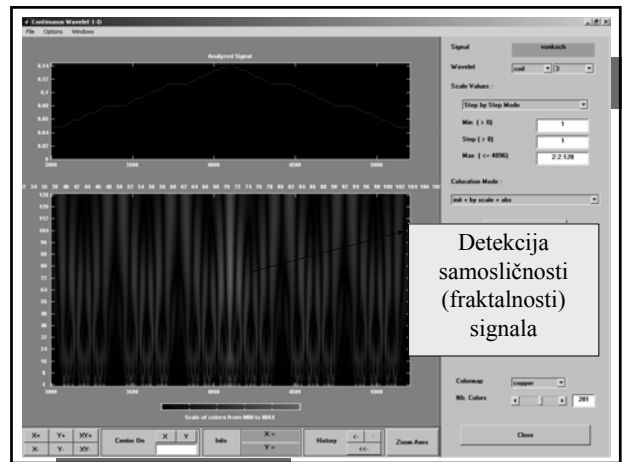
DWT wavelet obitelji

- Za primjene su vrlo značajni filtri s konačnim impulsnim odzivom (FIR).
- Obitelji konačnih waveleta:
 - Haar: 1. red, ortogonalni, simetrični, najstariji (1900.);
 - Daubechies: ortonormalni, maks. glatki, nesimetrični;
 - Symlets: ortonormalni, skoro simetrični;
 - Biortogonalni, simetrični (linearne faze), ...

DWT primjene

- Potiskivanje šuma
- Kompresija
 - 1D signala
 - audio & govorni signali,
 - EKG signali, ...
 - 2D slika i video signala,
 - otisaka prstiju (FBI), ...
- Ekstrakcija značajki signala
- ...





Dodatne teme

- Lifting shema (kratko u dodatnoj literaturi)
- 2D poopćenje (u nastavku prezentacije):
 - separabilno,
 - neseparabilno.
- 2D primjeri
- Adaptivni waveleti:
 - 1D (u dodatnoj literaturi),
 - 2D (u nastavku prezentacije).

Wavelets

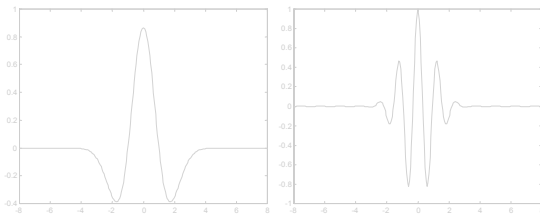
Problems of 2-D generalization of wavelet filter banks with variant number of zero moments

Miroslav Vrankić, dipl. ing. FER, ZESOI, 2001. 11. 8.

Discrete wavelet transform

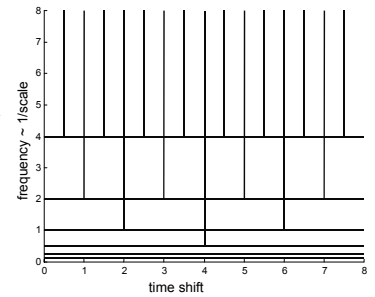
Some typical wavelets

- Mexican hat
- Morlet wavelet



DWT properties

- localized base functions
- logarithmic frequency axis
- time-frequency tiling



Where to use wavelets??

- compression
- denoising
- communications

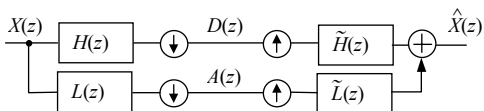
Where to use wavelets??

- JPEG 2000: wavelet-based image compression standard



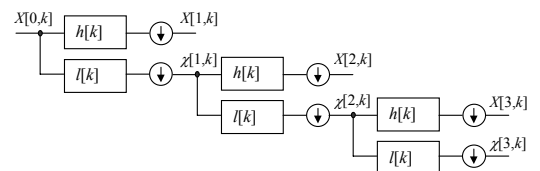
Reconstructed image after compression at 0.25 b/p using JPEG (left) and JPEG 2000 (right).

Wavelet filter banks



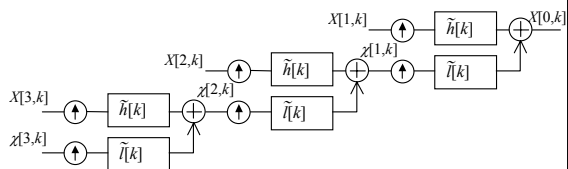
- perfect reconstruction: $X(z) = \hat{X}(z)$

Multiresolution analysis



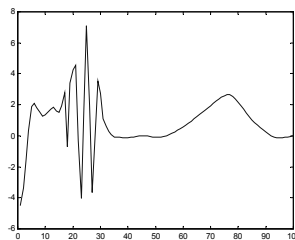
- convergency and regularity of elementary decomposition functions
 - $H(z)$ and $L(z)$: number of vanishing moments

Reconstruction

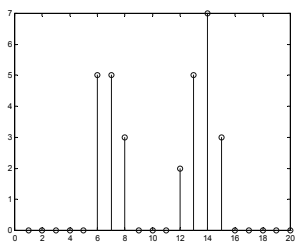


- every reconstruction level doubles the resolution

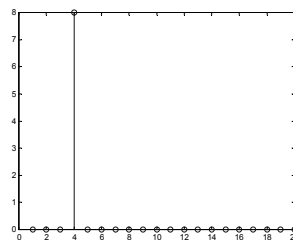
MRA example: input signal



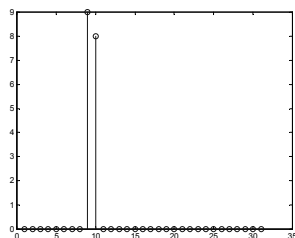
MRA example: level 3 average coefficients



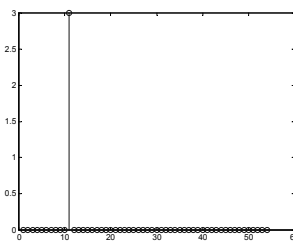
MRA example: level 3 difference coefficients



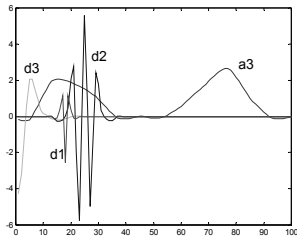
MRA example: level 2 difference coefficients



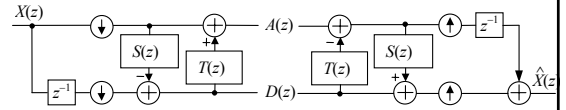
MRA example: level 1 difference coefficients



MRA example: reconstruction



Lifting scheme

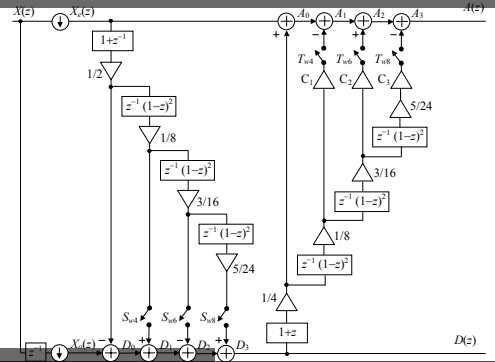


- $S(z)$ – predict stage
- $T(z)$ – update stage

Lifting scheme properties

- up to two times less computations
- in-place calculation
- easy to implement time-variant and non-linear PR filter banks
- possible to realize minimum delay filter banks

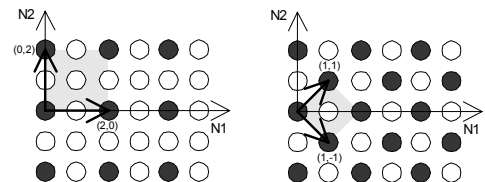
Adaptive number of vanishing moments



2-D wavelets bases

- separable
 - successive 1-D processing of rows and columns
- nonseparable
 - true multidimensional approach
 - higher computational complexity
 - greater flexibility in filter design
 - better adapted to human visual system

Sampling lattices



- separable

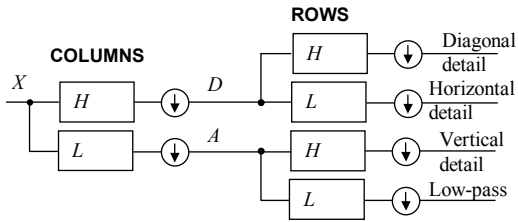
$$D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

- nonseparable (quincunx)

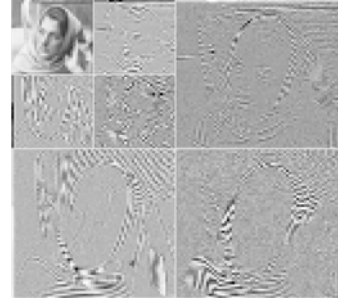
$$D = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Separable wavelet transform

- successive 1-D processing of rows and columns



Separable image decomposition



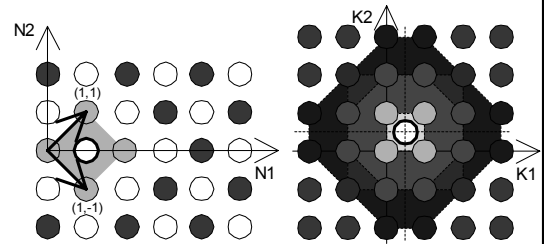
Quincunx

- quincunx decimation
- quincunx interpolating filter banks
- for example*:

*Jelena Kovačević, Wim Sweldens: Wavelet Families of Increasing Order in Arbitrary Dimensions

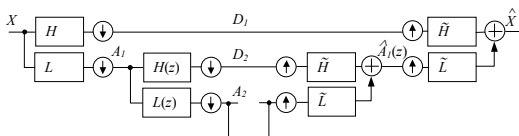
Quincunx interpolation

- increasing number of rings increases filter order



Quincunx decomposition

- multiresolution decomposition using 2-D filters



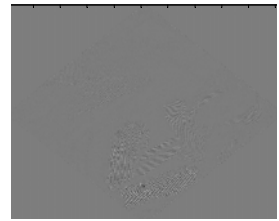
MRA example: original picture



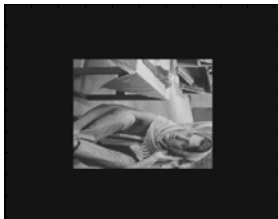
MRA example: A1 coefficients



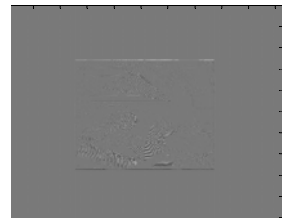
MRA example: D1 coefficients



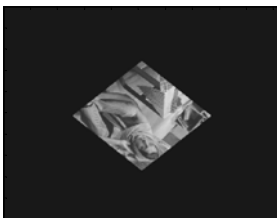
MRA example: A2 coefficients



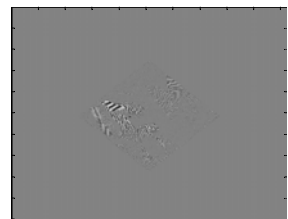
MRA example: D2 coefficients



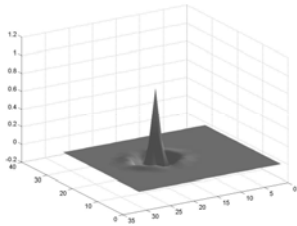
MRA example: A3 coefficients



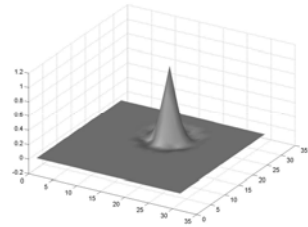
MRA example: D3 coefficients



Wavelet



Scaling function



Conclusion

- nonseparable filters are preferred
- 2D filters with variable parameters are to be implemented
 - generalisation of adaptive 1-D wavelet filter banks
- promising results

2-D Nonseparable Wavelet Filter Bank with Adaptive Filter Parameters



Damir Seršić
Miroslav Vrankić

Outline of the presentation

- Introduction
- Lifting scheme
- Quincunx 2D generalization
- 2D adaptive filter bank
- Adaptation of filter parameters
- Examples and results
- Conclusion

Introduction

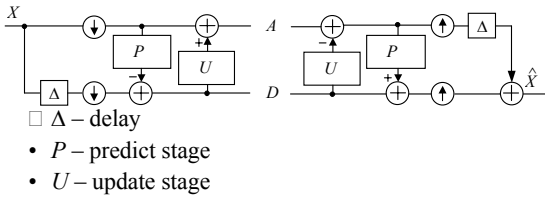
- Proposal: non-separable 2D wavelet filter bank with adaptive filter parameters.
- 2D generalization of previously reported 1D wavelet filter bank with adaptive filter parameters.

FOR MORE INFO...

D. Seršić: "A realization of wavelet filter bank with adaptive filter parameters,"
Proc. of EUSPICO 2000, 10(3):1733-1736

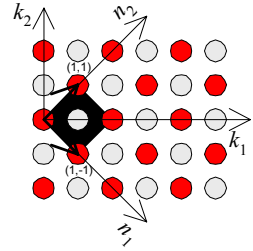
Lifting scheme

- An efficient way of construction of perfect reconstruction filter banks.



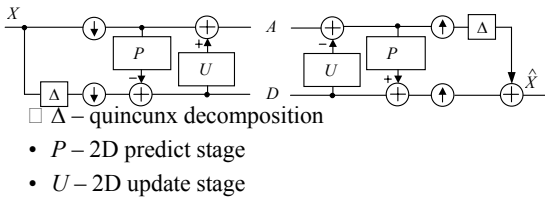
2D generalization

- Separable pyramid scheme:
 - 1D filters,
 - 4 cosets.
- Non-separable quincunx scheme:
 - 2D filters,
 - 2 cosets.



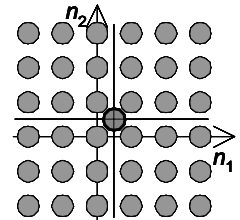
2D quincunx lifting scheme

- The same construction of 2D perfect reconstruction filter banks.



Interpolating prediction filters

- Neville interpolating filters
- Symmetric interpolation neighborhoods



FOR MORE INFO...

Kovačević J. and Sweldens W., Wavelet Families of Increasing Order in Arbitrary Dimensions, IEEE Trans. on Image Proc, 9(3):480-496, 2000

Examples of linear prediction

- 1D linear prediction:

$$P_2(z) = \frac{1}{2}(1 + z^{-1})$$

- 2D linear prediction:

$$P_2(z_1, z_2) = \frac{1}{4}(1 + z_1^{-1} + z_2^{-1} + z_1^{-1} z_2^{-1})$$

- etc.
- Update stage: $U_n = 1/2 P_n$.

The idea of adaptation

$$P = p_1 P_2 + p_2 (P_4 - P_2) + p_3 (P_6 - P_4) + p_4 (P_8 - P_6) + \dots$$

- Adaptive prediction parameters p_1, p_2, p_3, \dots
- If successively set to value one: fixed interpolating wavelets with desired number of vanishing moments N .
- Easy split in fixed and variable part.
- Example of the HP filter with two vanishing moments and one variable parameter:
 $p_1 = 1$, variable $p_2, p_3 = p_4 = \dots = 0$.

The idea of adaptation

$$U = u_1 U_2 + u_2 (U_4 - U_2) + u_3 (U_6 - U_4) + \dots$$

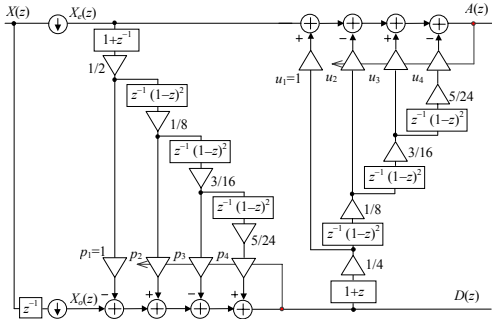
- Adaptive update parameters u_1, u_2, u_3, \dots
- Parameters u_m can be chosen to achieve desired number of vanishing moments M of the LP filter.
- Of course, values of u_m are dependant on the previous stage parameters p_n , but...

Dependence of filter parameters

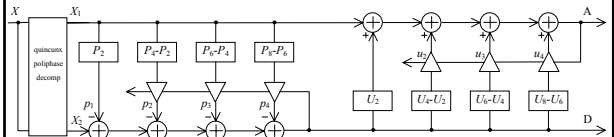
- If $N \geq M$, parameters u_m should be set to value one in order to achieve zero moments.
- Colored values are valid for 1D banks only.
- Moreover, moments are preserved for any value of variable parameters $p_n \neq 0$ and $u_m \neq 0$!

$\{p_1, p_2, p_3, p_4\}$	$\{1, 0, 0, 0\}$	$\{1, 1, 0, 0\}$	$\{1, 1, 1, 0\}$	$\{1, 1, 1, 1\}$
u_1	1	1	1	1
u_2	3/2	1	1	1
u_3	5/3	3/2	1	1
u_4	7/4	3/2	3/2	1

1D adaptive wavelet filter bank



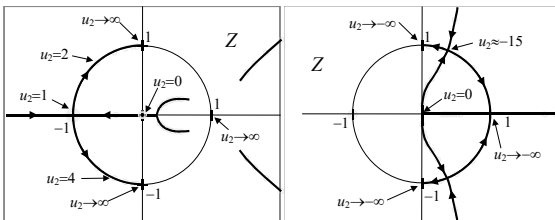
2D adaptive wavelet filter bank



- Example with 2+2 vanishing moments and 3+3 variable parameters.
- We have not found similar factorization of P and U stages as we did for 1D filters.

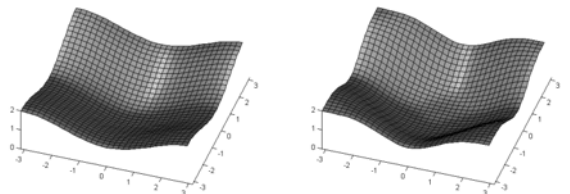
Back to 1D example

- 2+2 vanishing moments, variable u_2 .
- Zero locus plots of the LP filter:

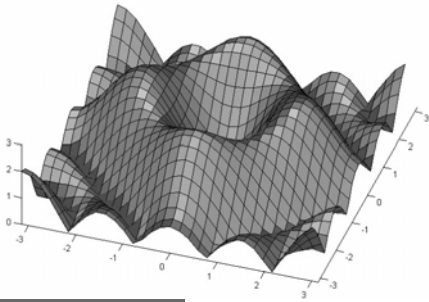


2D example

- 2+2 vanishing moments, variable p_2 .
- HP filter frequency responses for $p_2 = 0$; 1.



HP filter frequency responses



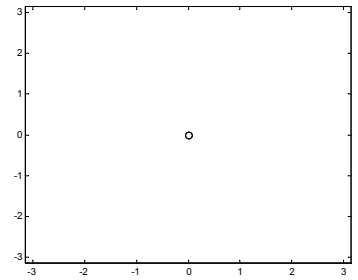
Zeros of the HP frequency response

- negative values of p_2

◆ $p_2 = -500$

◆ $p_2 = -10$

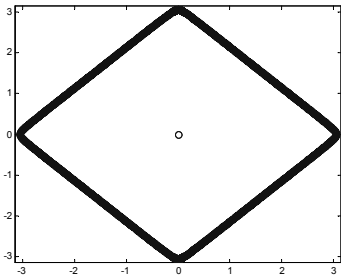
◆ $p_2 = -1$



Zeros of the HP frequency response

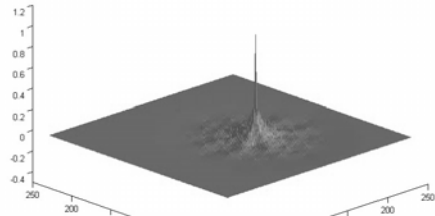
- Non-negative values of p_2

- ◆ $p_2 = 0$
- ◆ $p_2 = 1$
- ◆ $p_2 = 2$
- ◆ $p_2 = 10$
- ◆ $p_2 = 500$



Limit wavelet functions

$p_2 = -3$



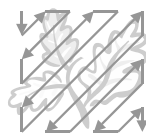
Adaptation of the filter parameters

- 1D Recursive Least Squares (RLS).
- Adapts filter parameters in order to minimize wavelet coefficients.
- Snake-like direction of adaptation:

vertical

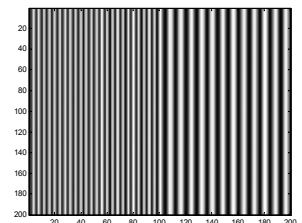


diagonal



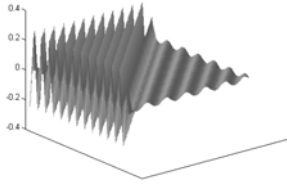
Analysis of a synthetic signal

- Two horizontal sine waves
 $\omega = \pi/2; \pi/4$



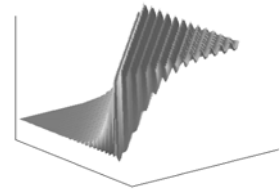
Fixed wavelet decomposition

- Wavelet D coefficients for fixed P_4 .



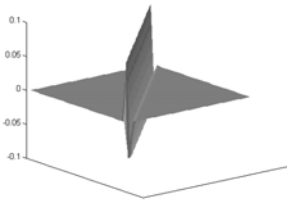
Adaptive wavelet decomposition

- Fixed 2+2, RLS adaptive p_2 , $\lambda = 0.85$.



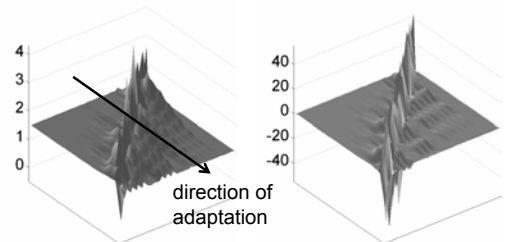
Adaptive wavelet decomposition

- D coefficients with adapted $p_2 = 2$; 1.17.



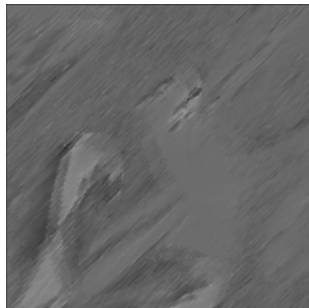
Adaptive wavelet decomposition

- Adapted p_2 ($\lambda = 0.82$): D coefficients:



Adaptation using other 1D methods

- 1D LSW
- Window 9 pixels wide
- Diagonal orientation
- Visible directional artifacts



Discussion

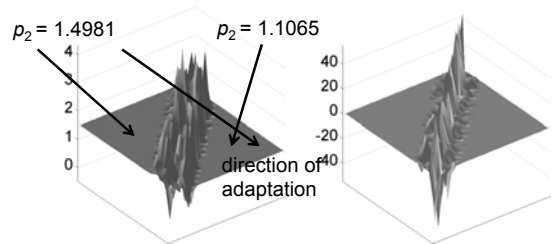
- Image information is divided between wavelet coefficients and filter parameters.
- Wavelet coefficients have lower values and lower entropy when compared to fixed FB-s.
- Filter parameters can change slowly ($>\lambda$).
- Causal adaptive algorithms do not require separate transmission of the filter parameters.
- 1D algorithms are directionally dependent.
- 2D adaptation algorithms should have better behavior (LSW, robust LSW, BLUE, ...).

1-D LSW

- windowed least squares
- window sizes: $N = 3, 6, 9$ or 12 samples
- adapt. area lies in a line
- over determined set of linear equations

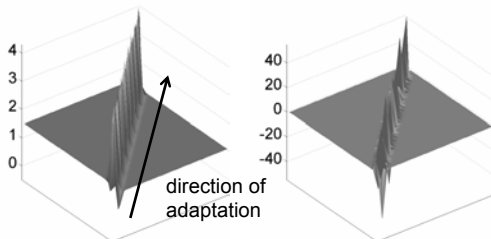
1-D LSW with vertical orientation

- adapted p_2 for $N = 6$:
- resulting D image:



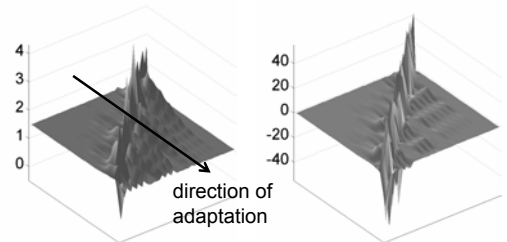
1-D LSW with diagonal orientation

- adapted p_2 for $N = 6$:
- resulting D image:



1-D Recursive LS

- adapted p_2 for $\lambda = 0.82$:
- resulting D image:

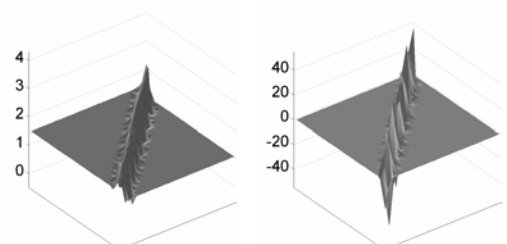


2-D LSW

- 2-D windowed least squares
- more appropriate than 1-D approach
- adaptation window
 - surrounds adaptation point
 - rectangular
 - 3x3 or 5x5 pixels

2-D LSW

- p_2 for 3x3 window used:
- resulting D image:

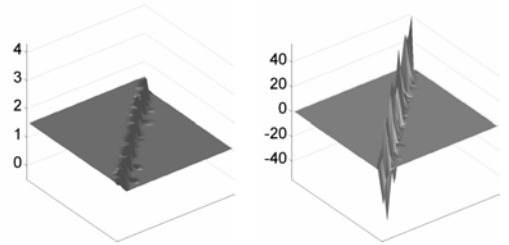


Robust 2-D LSW

- modification of 2-D LSW
- reduces transition area
- traces prevalent image characteristics
- choose only M samples inside window...
- that have least quadratic prediction error
- iterative method

Robust 2-D LSW

- p_2 for $M = 10$ on 5×5 window:
- resulting D image:



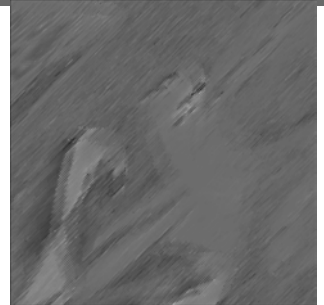
Real-world Test Image

- test image Barbara
- first level D with fixed $p_2 = 0$ ($P = P_2$)



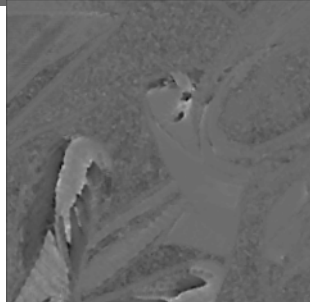
Adapted p_2 Parameters

- using 1-D LSW
- window 9 pixels wide
- diagonal orientation



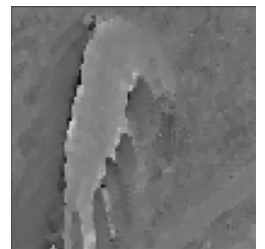
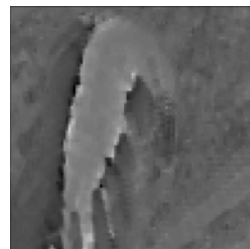
Adapted p_2 Parameters

- using 2-D LSW
- 3×3 pixels window



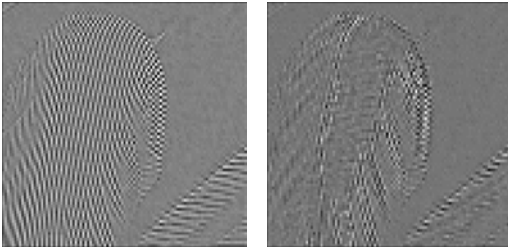
Adapted p_2 Parameters

- 2-D LSW
- window: 3×3
- robust 2-D LSW
- window: 3×3 , $M = 5$



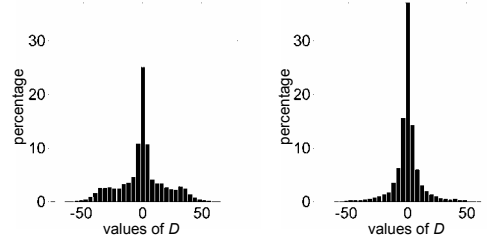
Detail Coefficients

- no adaptation
- robust 2-D LSW
- $p_2 = 0$ ($P = P_2$)
- window: 3×3 , $M = 5$



Detail Coefficients' Histograms

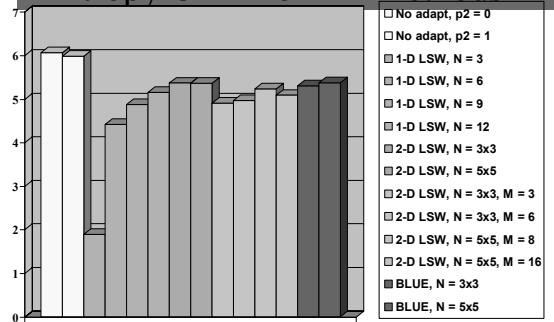
- no adaptation
- robust 2-D LSW
- $p_2 = 0$ ($P = P_2$)
- window: 3×3 , $M = 5$



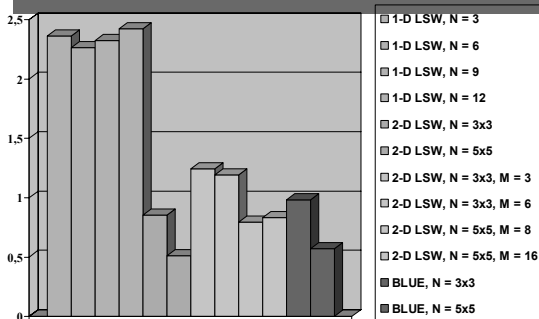
BLUE Method

- Best Linear Unbiased Estimate
- makes adaptation spectrally dependent
- keeping filter parameters in admissible range
- 3-diagonal weight matrix

Entropy Of D For All Methods

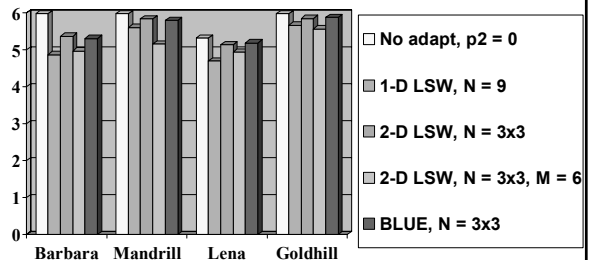


Spectrum Bandwidth of p_2



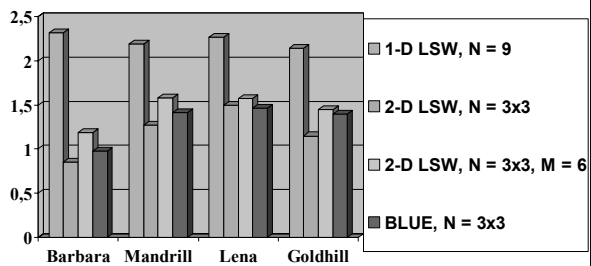
Some Real-world Images

Entropy of D coefficients



Some Real-world Images

Spectrum bandwidth of p_2



Conclusion



- Basic question: why to adapt?
- Should we analyze this image using a set of circles, rectangles, “circle-tangles” or ...
- using adaptive sets of functions (circles or rectangles, where needed).
- Presented ideas are just a step in that direction.