Fourierov spektar signala

Fourierov red

- Složeni periodički signal :

$$\tilde{x}(t) = \tilde{x}(t + nT), \quad n \in \mathbb{Z}$$

može se aproksimirati trigonometrijskim polinomom:

sinteza: $$\tilde{x}(t) = \sum_{-N}^{N} a_n e^{j\overline{\omega}_1 nt}, \quad n \in \mathbb{Z}$$

odnosno sumom eksponencijala.

- Za realni $\tilde{x}(t)$ kompleksne amplitude su konjugirane.

Fourierov red (nastavak)

$$a_{-n} = a_n^*; \quad a_n = A_n e^{j\varphi_n}, a_n^* = A_n e^{-j\varphi_n}$$

$$\tilde{x}_n(t) = A_n (e^{j\varphi_n} e^{j\overline{\omega}_1 nt} + e^{-j\varphi_n} e^{-j\overline{\omega}_1 nt})$$

$$\tilde{x}_n(t) = 2A_n \cos(\overline{\omega}_1 nt + \varphi_n)$$

- Koeficijenti a_n Fourierovog reda obično se određuju tako da se gornji red pomnoži s $e^{-jn\overline{\omega}_1 t}$ i integrira u osnovnom periodu T.
- Odatle izlazi Fourierov koeficient a_n

analiza: $$a_n = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jn\overline{\omega}_1 t} dt = X[n]$$

Svojstva Fourierovog Reda

$$\tilde{x}(t) = \sum X[n] e^{jn\omega_0 t}$$

$$\tilde{x}(t) \leftrightarrow X[n] \qquad \tilde{x}(t) \leftrightarrow X(n\omega_0) \qquad \omega_0 = \frac{2\pi}{T}$$

$$a\tilde{u}(t) + b\tilde{v}(t) \leftrightarrow aU(n\omega_0) + bV(n\omega_0)$$

$$a\tilde{u}(t) + b\tilde{v}(t) \leftrightarrow aU[n] + bV[n]$$

$$x(t+\tau) \leftrightarrow X[n] e^{jn\omega_0 t}$$

$$x(t) e^{jm\omega_0 t} \leftrightarrow X[n+m]$$

Svojstva Fourierovog Reda (nastavak)

$$\tilde{y}(t) = \frac{1}{T} \tilde{f}(t) * \tilde{g}(t) \leftrightarrow F[n] \cdot G[n]$$

$$Y[n] = F[n] * G[n] \leftrightarrow \tilde{f}(t) \cdot \tilde{g}(t)$$

$$\tilde{y}(t) = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{u}(t) \cdot \tilde{v}(t-\tau) d\tau \leftrightarrow \tilde{u} * \tilde{v}$$

cirkularna konvolucija

$$Y[n] = U[n] * V[n] \leftrightarrow \tilde{u}(t) \cdot \tilde{v}(t)$$

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |\tilde{x}(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |X[n]|^2$$

Poopćenje Fourierovog reda

- Elementarni signali u Fourierovom redu su eksponencijale, koje zadovoljavaju uvjet ortogonalnosti

$$\int_{-T/2}^{T/2} e^{jn\overline{\omega}_1 t} e^{-jm\overline{\omega}_1 t} dt = \begin{cases} T, & m = n \\ 0, & m \neq n \end{cases}$$

Napisano općenito za vremenski kontinuirane i diskretne signale

$$\int_{-T/2}^{T/2} \varphi_n(t) \varphi_m^*(t) dt = \begin{cases} K_n, & m = n \\ 0, & m \neq n \end{cases}$$

$$\sum_{0}^{K-1} \varphi_n(k) \varphi_m^*(k) = \begin{cases} K_n, & m = n \\ 0, & m \neq n \end{cases}$$

Poopćenje Fourierovog reda (nastavak)

- Pri predstavljanju složenih signala linearnom kombinacijom elementarnih signala, često se upotrebljavaju ortogonalne funkcije.

$$x(t) \cong \sum_{n=0}^{N} a_n \varphi_n(t) \qquad \text{ili} \qquad x[k] \cong \sum_{n=0}^{N} a_n \varphi_n[k]$$

- Koeficijenti a_n mogu se odrediti na temelju minimalne greške aproksimacije. Pogodna karakterizacija greške je integral ili suma kvadrata greške u danom intervalu.

$$\varepsilon = \frac{1}{k_1 - k_2} \sum_{k_1}^{k_2} \left[x[k] - \sum_{n=1}^{N} a_n \varphi_n[k] \right]^2$$

Poopćenje Fourierovog reda (nastavak)

- Nađimo optimalne koeficijente a_1 i a_2 traženjem minimuma greške

$$\frac{\partial \varepsilon}{\partial a_1} = 0, \quad \frac{\partial \varepsilon}{\partial a_2} = 0$$

$$\varepsilon = \sum_k \left\{ x^2[k] - 2x[k]\left[a_1\varphi_1[k] + a_2\varphi_2[k]\right] + \right.$$

$$\left. + \left[a_1\varphi_1[k] + a_2\varphi_2[k]\right]^2 \right\}$$

- Pri kvadriranju sumacije otpadaju miješani članovi zbog ortogonalnosti, tako da izlaze uvjeti ekstrema:

Poopćenje Fourierovog reda (nastavak)

$$-2x[k]\varphi_1[k] + 2a_1\varphi_1^2[k] = 0$$

$$-2x[k]\varphi_2[k] + 2a_2\varphi_2^2[k] = 0$$

- Odakle izlaze optimalni koeficijenti a_1 i a_2

$$a_1 = \frac{\sum x[k]\varphi_1[k]}{\sum \varphi_1^2[k]} = \frac{\alpha_1}{K_1}$$

$$a_2 = \frac{\sum x[k]\varphi_2[k]}{\sum \varphi_2^2[k]} = \frac{\alpha_2}{K_2}$$

Poopćenje Fourierovog reda (nastavak)

- Kvadratna greška aproksimacije konačnom sumom do N

$$\varepsilon = \frac{1}{k_2 - k_1} \sum_{k_1}^{k_2} \left[x[k] - \sum_{n=1}^{N} a_n \varphi_n[k] \right]^2 =$$

$$= \frac{1}{k_2 - k_1} \sum_{k_1}^{k_2} \left[x^2[k] + \sum_n a_n^2 \varphi_n^2 - 2x \sum_n a_n \varphi_n[k] \right] =$$

$$= \frac{1}{k_2 - k_1} \left[\sum_{k_1}^{k_2} x^2[k] + \sum_n a_n^2 K_n - 2 \sum_n a_n \alpha_n \right]$$

Poopćenje Fourierovog reda (nastavak)

- ako nadopunimo desne članove s $+\dfrac{\alpha_n^2}{K_n}$

funkcija kvadrat $\left(a_n^2 K_2 - 2a_n\alpha_n + \dfrac{\alpha_n^2}{K_n} \right) - \dfrac{\alpha_n^2}{K_n}$

izlazi $\left(a_n\sqrt{K_n} - \dfrac{\alpha_n}{\sqrt{K_n}} \right)^2 - \dfrac{\alpha_n^2}{K_n}$

Poopćenje Fourierovog reda (nastavak)

- Budući da za optimalne koeficijente vrijedi $a_n = \alpha_n / K_n$ najmanja greška je dana s

$$\varepsilon = \sum_k x^2[k] - \sum_n \frac{\alpha_n^2}{K_n}$$

odnosno zbog $\alpha_n^2 / K_n = a_n^2 K_n$

$$\varepsilon = \sum_{k_1}^{k_2} x^2[k] - \sum_{1}^{N} a_n^2 K_n$$

- kako su sumandi nenegativni može se zaključiti da s većim N greške aproksimacije su sve manje.

Poopćenje Fourierovog reda (nastavak)

- Kad N raste bez granica suma () $\sum_n a_n^2 K_n$ konvergira sumi $\sum_k x^2[k]$ što predstavlja energiju signala.
- U tom slučaju vrijedi

$$\sum_{k=1}^{K} x^2[k] = \sum_{1}^{N} a_n^2 K_n$$

što je generalizirani oblik Parsevalove relacije.

Poopćenje Fourierovog reda (nastavak)

Ako vrijedi za neki niz $x[k]$ kaže se da suma $\sum_n a_n \varphi_n[k]$ u prosjeku konvergira nizu $x[k]$.

Vremenski diskretni Fourierov red (DFT)

Uzmimo periodičan niz za koji vrijedi

$$\tilde{x}[k] = \tilde{x}[k + Nr], \quad r \in \mathbb{Z}$$

Kao kod kontinuiranog periodičnog signala može se razložiti na sumu periodičkih sinusoida ili eksponencijala frekvencija koje su cjelobrojni višekratnici osnovne $2\pi/N$

$$g_n[k] = e^{\frac{2\pi}{N}kn} = e^{\frac{2\pi n}{N}[k+rN]} = e_n[k + rN]$$

Vremenski diskretni Fourierov red (DFT) (nastavak)

Budući da su eksponencijale diskretne najviša frekvencija koja se može jednoznačno predstaviti je s $n=N-1$. Sve ostale $n \geq N$ mogu se naći među onima iz intervala $[0, N-1]$. Među svim eksponencijalama perioda N mogu se dakle naći samo N različitih

$$g_0, g_1, \ldots, g_{N-1}$$

jer: $g_0[k] = g_N[k],\ g_1[k] = g_{N+1}[k], \ldots$

Vremenski diskretni Fourierov red (DFT) (nastavak)

Periodičan niz $\tilde{x}[k]$ dakle se može predstaviti s N diskretnih eksponencijala

$$\tilde{x}[k] = \sum_{n=0}^{N-1} a_n e^{j\frac{2\pi}{N}nk}$$

Optimalni koeficijenti koji osiguravaju minimum sume kvadrata greške mogu se dobiti iz općeg izraza za razlaganje signala na ortogonalne nizove.

Vremenski diskretni Fourierov red (DFT) (nastavak)

Optimalni koeficijenti su:

$$a_n = \frac{\sum_{0}^{N-1} x[n]\varphi_n^*[k]}{\sum_{0}^{N-1} \varphi_n[k]\varphi_n^*[k]} = \frac{\sum_{0}^{N-1} x[k] e^{-\frac{2\pi nk}{N}}}{N}$$

Vremenski diskretni Fourierov red (DFT) (nastavak)

$$\widetilde{X}[n]=\frac{1}{N}\sum_{0}^{N-1}\widetilde{x}[k]e^{-\frac{2\pi nk}{N}}\ ;\ \ \widetilde{x}[k]=\sum_{n=0}^{N-1}\widetilde{X}[n]e^{j\frac{2\pi}{N}nk}$$

čine par izraza koji se nazivaju diskretnom Fourierovom transformacijom (DFT)

Kako se vidi niz koeficijenata a_n je također periodičan niz tj. $a_n=a_{n+N}=\widetilde{X}[n]$

s periodom N: $e^{j2\pi kn/N}\cdot e^{j2\pi rNn/N}=e^{j2\pi kn/N}$.

DFT povezuje N uzoraka jednog perioda periodičkog signala s N uzoraka periodičkog spektra.

Koeficijent $1/N$ se nekad pridružuje izrazu za $\widetilde{x}[k]$.

Vremenski diskretni Fourierov red (DFT) (nastavak)

Pogreška aproksimacije

Suma kvadrata greški VDFR-a ili DTFT-a se može dobiti iz općeg izraza () i $K_n=N$

$$\sum_{k=0}^{N-1}e^{j\frac{2\pi}{N}k(n-m)}=\begin{cases}N, & n=m\\ 0, & n\neq m\end{cases}$$

$$\varepsilon=\sum_{k=0}^{N-1}x^2[k]-\sum_{n=0}^{N-1}a_n^2N=0.$$

Vremenski diskretni Fourierov red (DFT) (nastavak)

$$X[n]=\sum_{k=0}^{N-1}x[k]e^{-j\frac{2\pi nk}{N}}=\sum_{k=0}^{N-1}x[k]W^{nk}$$

$$x[k]=\sum_{k=0}^{N-1}X[n]e^{j\frac{2\pi nk}{N}}=\sum_{k=0}^{N-1}X[n]W^{-nk}$$

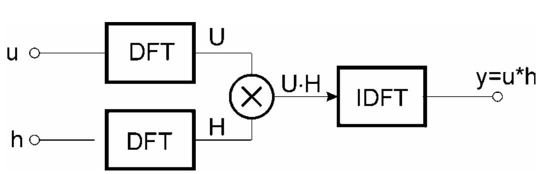
$$W=e^{-j\frac{2\pi}{N}}.$$

Periodičnost niza i periodičnost spektra

- Izvan područja $n\in[0, N\text{-}1]$ se nizovi signala i spektra ne ponavljaju.
- Izraz $[k\text{-}i]\text{mod}N$ znači da $[k\text{-}i]$ treba dijeliti s N i sačuvati samo ostatak.
- Da bi se preko cirkularne konvolucije stiglo na linearnu trebat će nadopuniti impulsima oba niza tako da period bude jednak cirkularnoj dužini konvolucije.
- Za slučaj dužine sekvencije M i N konvolucija će biti dužine $M+N\text{-}1$.

Periodičnost niza i periodičnost spektra (nastavak)

- Odziv sustava kao linearna konvolucija traži više multiplikacija nego pretvorba u spektar oba signala pobude i odziva na uzorak množenjem spektara i inverzijom.



Svojstva DTFS (DFT)

Linearnost

$$\text{DTFS}\{a\widetilde{u}[k]+b\widetilde{v}[k]\}=a\widetilde{U}[n]+b\widetilde{V}[n]$$

Posmak

$$\{\widetilde{x}[k-i]\text{mod}\,N\}\leftrightarrow\widetilde{X}[n]e^{-j\frac{2\pi ki}{N}}$$

Konvolucija cirkularna

$$\sum_{i=0}^{N-1}\widetilde{u}[k-i]\text{mod}\,N\widetilde{v}[i]\leftrightarrow\widetilde{U}[n]\cdot\widetilde{V}[n]$$

Parseval

$$\sum_{k=0}^{N-1}\left|\widetilde{x}[k]\right|^2=\sum_{n=0}^{N-1}\left|\widetilde{X}[n]\right|^2$$

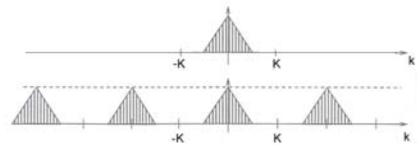
Vremenski diskretna Fourierova transformacija

Jednostavan prijelaz iz Fourierovih redova u transformaciju može se dobiti prikazom aperiodičkog signala kao graničnog slučaja periodičkog signala kad period ponavljanja teži u beskonačnost.

Vremenski diskretna Fourierova transformacija (nastavak)

Pretpostavimo da je aperiodički signal $x[k]$ dan jednim periodom signala $\tilde{x}[k]$, koji je periodičan s N.

$$x[k] = \begin{cases} \tilde{x}[k], & -K \le k \le K \\ 0, & |k| \ge K \end{cases} \qquad N = 2K + 1$$



Vremenski diskretna Fourierova transformacija (nastavak)

Kad K raste, razmak između sekcija signala se povećava, te za $K \to \infty$ replike se udaljavaju u beskonačnost.

$$x[k] = \lim_{K \to \infty} \tilde{x}[k]$$

Vremenski Diskretni Fourierov red periodičkog niza $\tilde{x}[k]$ je:

$$\tilde{x}[k] = \sum_{n=-K}^{K} \tilde{X}[n] e^{j\frac{2\pi kn}{N}} \qquad \tilde{X}[n] = \frac{1}{N} \sum_{n=-K}^{K} \tilde{x}[k] e^{-j\frac{2\pi kn}{N}}$$

Vremenski diskretna Fourierova transformacija (nastavak)

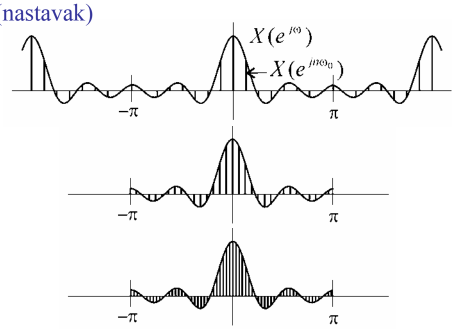
Budući da je $x[k] = \tilde{x}[k]$ za $-K \le k \le K$ i $x[k] = 0$ za $k > K$ izlazi

$$X[n] = \frac{1}{N} \sum_{n=-K}^{K} x[k] e^{-j\frac{2\pi nk}{N}} =$$

$$= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[k] e^{-j\frac{2\pi nk}{N}}$$

$X[n]$ je periodičan s N. Spektralni uzorci su s razmakom $2\pi / N = \omega_0$

Vremenski diskretna Fourierova transformacija (nastavak)



Slika prikazuje uzorke spektra i njihovu ovojnicu, koja je periodična s 2π. Za veći N uzorci postaju sve gušći.

Vremenski diskretna Fourierova transformacija (nastavak)

Zamislimo da tom nizu uzoraka na slici spektra odredimo normiranu ovojnicu kontinuiranu periodičku funkciju, tako da vrijedi

$$X[n] = \frac{1}{N} X(e^{j\omega})\Big|_{\omega = n\omega_0}$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k}$$

$$X(e^{j\omega})\Big|_{\omega = n\omega_0} = \sum_{k=-\infty}^{\infty} x[k] e^{-jn\omega_0 k}$$

Vremenski diskretna Fourierova transformacija (nastavak)

Sad se periodički spektar može izraziti s normiranom ovojnicom

$$\tilde{x}[k]=\frac{1}{N}\sum_{n=-K}^{+K}X(e^{j\omega_0 n})e^{j\omega_0 nk}$$

$$x[k]=\frac{\omega_0}{2\pi}\sum_{n=-K}^{+K}X(e^{j\omega_0 n})e^{j\omega_0 nk}$$

Utjecaj graničnog prijelaza $N\to\infty$ je da smanjuje razmak ω_0 između komponenti spektra $\omega_0=2\pi/N$.

Vremenski diskretna Fourierova transformacija (nastavak)

U sumaciji imamo vrijednosti $X(e^{j\omega_0 k})\cdot e^{j\omega_0 k}$ množene sa širinom $\omega_0=2\pi/N$. Sumacija je pravokutna aproksimacija integrala.

Kad $N,K\to\infty$, $\omega=k\omega_0$, $d\omega=\omega_0$, a suma prelazi u integral

$$x[k]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})\cdot e^{j\omega k}d\omega$$

Vremenski diskretna Fourierova transformacija (nastavak)

Time smo dobili aperiodički niz $x[k]$ kao superpoziciju eksponencijala ili sinusoida. Težinska funkcija je spektar

$$X(e^{j\omega})=\sum_{k=-\infty}^{\infty}x[k]\cdot e^{-j\omega k}$$

pa zajedno sa integralom čini par koji se naziva vremenski diskretnom Fourierovom transformacijom VDFT (engl. Discrete Time Fourier Transform DTFT).

Vremenski diskretna Fourierova transformacija (nastavak)

Uvjet da aperiodički niz ima DTFT je da njegova sumacija apsolutno konvergira

$$\sum_{-\infty}^{\infty}|x[k]|<\infty.$$

Svojstva DTFT

Linearnost

$$au[n]+bv[n]\leftrightarrow aU(e^{j\omega})+bV(e^{j\omega})$$

Posmak

$$x[k-i]\leftrightarrow e^{-ji\omega}X(e^{j\omega})$$

Konvolucija

$$u*v=\sum_{i=-\infty}^{\infty}u[i]v[k-i]\leftrightarrow U(e^{j\omega})\cdot V(e^{j\omega})$$

Svojstva DTFT (nastavak)

Parseval

$$\sum_{i=-\infty}^{\infty}x[k]x^*[k]=\sum_{i=-\infty}^{\infty}|x[k]|^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\omega})\right|^2d\omega$$

Množenje

$$u[k]v[k]=\frac{1}{2\pi}\int_{-T/2}^{T/2}U(e^{j\Theta})V(e^{j(\omega-\Theta)})d\Theta$$ periodična konvolucija

Fourierova transformacija

Upotrebljava se za predstavljanje aperiodskih signala superpozicijom eksponencijala ili sinusoida.
Može se izvesti iz Fourierovog reda, tako da se aperiodski signal dobije kao granični slučaj periodičnog signala, čiji period ide u beskonačnost.
Slično kao kod DTFT

$$x(t)=\begin{cases}\tilde{x}(t), & -T/2\le t\le T/2\\ 0, & |t|>T/2\end{cases} \qquad x(t)=\lim_{T\to\infty}\tilde{x}(t)$$

Fourierova transformacija (nastavak)

Harmonijske komponente postaju guste, pa dobivamo iz sume integral:

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\varpi)\cdot e^{j\varpi t}d\varpi$$

$$X(j\varpi)=\int_{-\infty}^{\infty}x(t)\cdot e^{-j\varpi t}dt$$

Fourierov spektar signala

Spektar signala napisan u pravokutnom obliku sa svojim realnim i imaginarnim dijelom

$$X(j\omega)=X_r(j\omega)+jX_i(j\omega)$$

napisan u polarnom obliku sa svojim amplitudnim i faznim spektrom

$$X(j\omega)=|X(j\omega)|e^{j\varphi(\omega)}$$

$$|X(j\varpi)|=A(\varpi), \qquad X_r(\varpi)=A(\omega)\cos\varphi(\varpi)$$

$$\varphi(\omega)=arctg\frac{X_i(\omega)}{X_r(\omega)}, \quad X_i(\varpi)=A(\omega)\sin\varphi(\varpi)$$

Fourierov spektar signala (nastavak)

Da bi frekvencija signala imala Fourierovu transformaciju mora zadovoljavati neke uvjete:

$$1.\ \int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt\le\int_{-\infty}^{+\infty}\left|x(t)e^{-j\omega t}\right|dt\le\int_{-\infty}^{+\infty}|x(t)|dt<\infty$$

Funkcija $x(t)$ mora biti apsolutno integrabilna te imati konačan broj maksimuma i minimuma, tj. konačan broj diskontinuiteta u konačnom intervalu.

Fourierov spektar signala (nastavak)

Transformacija postoji za praktički upotrebljive signale. Ima međutim signala kao što su stepenica i sinusoida koje nisu apsolutno integrabilne, ali se mogu predstaviti transformacijom, ako dozvolimo upotrebu impulsa u vremenskom i frekvencijskom domenu.

Fourierov spektar signala (nastavak)

Primjer: Spektar pravokutnog pulsa.

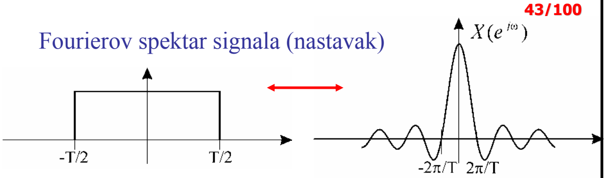
Pravokutni puls je definiran:

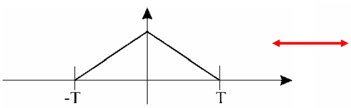
$$r_C(t)=\begin{cases}1, & \text{za } |t|<T/2\\ 0, & \text{za } |t|>T/2\end{cases}$$

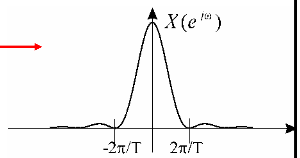
$$R_C(\omega)=\int_{-\infty}^{+\infty}r_C(t)e^{-j\varpi t}dt=\int_{-T/2}^{T/2}1\cdot e^{-j\varpi t}dt=\left.\frac{e^{-j\omega t}}{j\omega}\right|_{-T/2}^{T/2}=$$

$$=\frac{e^{-j\varpi T/2}-e^{j\varpi T/2}}{j\varpi t}=T\frac{\sin(\varpi T/2)}{(\varpi T/2)}=T\mathrm{sinc}\,(\varpi T/2)$$

Fourierov spektar signala (nastavak)







Trokutni puls:

$$u_r(t)=\begin{cases} t-|t/T|, & \text{za } |t|\le T \\ 0, & \text{za } |t|>T \end{cases}$$

$$TR(\varpi)=T\,\mathrm{sinc}^2\,(\varpi T/2)$$

Fourierov spektar signala (nastavak)

Simetrija FT među varijablama t i ϖ omogućuju lagano određivanje odnosa između signala i spektra.

Ako je:

$$x(t)\longleftrightarrow X(j\varpi)$$

Tada je:

$$X(jt)\longleftrightarrow 2\pi x(-\varpi)$$

Dokaz slijedi iz izraza za $x(t)$ i zamjenom $t\rightarrow-\infty$

Prolaz signala kroz linearan sustav

- Kako smo ranije rekli sustav je skup operacija na ulaznom signalu da bi se dobio izlazni signal.
- Na temelju dosadašnjeg zaključujemo da se integralno vladanje sustava može odrediti iz njegovog odziva na impuls (KS) ili uzorak (DS) ili pak iz frekvencijske karakteristike.
- Prema tome za linearne sustave imamo još dva matematička modela.

Odziv na impuls ili na sinus pobudu

- Za razliku od mjerenja parametara sustava opisanog diferencijalnim jednadžbama, mjerenje impulsnog odziva ili frekvencijske karakteristike je dosta jednostavno.

Svojstva FT:	vrem. domena	frekv. domena
linearnost FT	$au(t)+bv(t)\leftrightarrow aU(\omega)+bV(\omega)$	
simetrija	$x(t)\leftrightarrow X(\omega)$	$X(t)\leftrightarrow 2\pi x(-\omega)$
kompresija expom	$x(at)=\frac{1}{\lvert a\rvert}X\left(\frac{\varpi}{a}\right)$	
konvolucija u VD	$x(t)\leftrightarrow X(\omega)$	$h(t)\leftrightarrow H(\omega)$
	$x*h\leftrightarrow X(\omega)\cdot H(\omega)$	
konvolucija u FD	$x\cdot h\leftrightarrow X(\omega)*H(\omega)$	
vremenski pomak	$x(t-t_0)\leftrightarrow X(\omega)e^{-j\omega t_0}$	

Svojstva FT (nastavak):	vrem. domena	frekv. domena
frekvencijski pomak	$x(t)e^{j\omega_0 t}\leftrightarrow X(\omega-\omega_0)$	
vremenska derivacija	$\frac{d^n x}{dt^n}\leftrightarrow (j\omega)^n X(\omega)$	
vremenska integracija	$\int_{-\infty}^{+\infty} x(\tau)d\tau\leftrightarrow \frac{X(\omega)}{j\omega}+\pi X(0)\delta(\omega)$	
frekvencijska derivacija	$-jtx(t)\leftrightarrow \frac{dX(\omega)}{d\omega}$	
frekvencijska integracija	$\frac{x(t)}{-jt}\leftrightarrow \int_{-\infty}^{+\infty} X(\Theta)d\Theta$	
vremenska inverzija	$x(-t)\leftrightarrow X(-\omega)$	

Transformacije:	vrem. domena	frekv. domena
	$\delta(t) \leftrightarrow 1$	
	$1 \leftrightarrow 2\pi\delta(\omega)$	
	$\cos\varpi t \leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
	$\sin\varpi t \leftrightarrow j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
	$\mathrm{sgn}\, t \leftrightarrow \frac{2}{j\pi}$	
	$u(t) \leftrightarrow \pi\delta(\omega)+\frac{1}{j\omega}$	

$$x(t)=\sum_{n=-\infty}^{+\infty} X_n e^{jn\omega_1 t} \leftrightarrow 2\pi \sum_{n=-\infty}^{+\infty} X_n \delta(\omega-n\omega_1)$$

Transformacije	vrem. domena	frekv. domena
(nastavak):	$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$	
	$\omega_0=2\pi/T$ $\sum_k \delta(t-kT) \leftrightarrow \omega_0 \sum_n \delta(\omega-n\omega_0)$	
	$\delta^{(n)}(t) \leftrightarrow (j\omega)^n$	
	$\lvert t \rvert \leftrightarrow -\frac{2}{\omega^2}$	
	$t^n \leftrightarrow 2\pi j^n \delta^{(n)}(\omega)$	

Četiri oblika Fourier-ovog predstavljanja signala

VD	periodički signal	aperiodički signal	VD / FD
kontinuirani signali	Fourierov red $x(t)=\sum_{n=-\infty}^{\infty} X[n]e^{jn\tilde{\omega}_0 t}$ $X[n]=\frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jn\tilde{\omega}_0 t}dt$ $x(t)$ ima period T $\quad \tilde{\omega}_0=\frac{2\pi}{T}$	Fourierova transformacija $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\tilde{\omega})e^{j\tilde{\omega}t}d\tilde{\omega}$ $X(j\tilde{\omega})=\int_{-\infty}^{\infty} x(t)e^{-j\tilde{\omega}t}dt$	aperiodički spektar
diskretni signali	Vremenski diskretan Fourierov red $x[k]=\sum_{n=\langle N\rangle} X[n]e^{jn\omega_0 k}$ $X[n]=\frac{1}{N}\sum_{k=\langle N\rangle} x[k]e^{-jn\omega_0 k}$ $x(k)$ i $X[n]$ imaju period N $\quad \omega_0=\frac{2\pi}{N}$	Vremenski diskretna Fourierova transformacija $x[k]=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega k}d\omega$ $X(e^{j\omega})=\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}$ $X(e^{j\omega})$ ima period 2π	periodički spektar
VD / FD	diskretni spektar	kontinuirani spektar	FD

VD-vremenska domena FD-frekvencijska domena

Papoulis

Time convolution theorem. The Fourier transform $F(\omega)$ of the convolution $f(t)$ of two functions $f_1(t)$ and $f_2(t)$ equals the product of the Fourier transforms $F_1(\omega)$ and $F_2(\omega)$ of these two functions. Thus if

$$f_1(t) \leftrightarrow F_1(\omega) \qquad f_2(t) \leftrightarrow F_2(\omega)$$

then

$$\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau \leftrightarrow F_1(\omega)F_2(\omega) \tag{2-71}$$

Proof. To prove (2-71), we shall form the Fourier integral of $f(t)$ and will show that it equals $F_1(\omega)F_2(\omega)$. Clearly

$$F(\omega)=\int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)\, d\tau\right] dt \tag{2-72}$$

Changing the order of integration, we obtain

$$F(\omega)=\int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} e^{-j\omega t} f_2(t-\tau)\, dt\right] d\tau$$

From the time-shifting theorem (2-36) we conclude that the bracket above equals $F_2(\omega)e^{-j\omega\tau}$; therefore

$$F(\omega)=\int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}F_2(\omega)\, d\tau = F_1(\omega)F_2(\omega)$$

and (2-71) is proved.

Comment. In the above proof it was assumed that the order of integration in (2-72) can be changed. This is true if the functions $f_1(t)$ and $f_2(t)$ are *square-integrable* in the sense

$$\int_{-\infty}^{\infty} |f_i(t)|^2\, dt < \infty \qquad i=1,2 \tag{2-73}$$

i.e., if $f_1(t)$ and $f_2(t)$ have finite energy.

Frequency convolution theorem. From the above result (2-71) and the symmetry property (2-34) it follows that the Fourier transform $F(\omega)$ of the product $f_1(t)f_2(t)$ of two functions equals the convolution $F_1(\omega) * F_2(\omega)$ of their respective transforms $F_1(\omega)$ and $F_2(\omega)$ divided by 2π:

$$f_1(t)f_2(t) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(y)F_2(\omega - y)\,dy \tag{2-74}$$

One could also give a direct proof of (2-74) as in the time-convolution theorem.

Parseval's formula. The following basic result, known as *Parseval's formula*, can be easily derived from (2-74); if $F(\omega) = A(\omega)e^{j\phi(\omega)}$ is the Fourier transform of $f(t)$, then

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} A^2(\omega)\,d\omega \tag{2-75}$$

Indeed, from $f(t) \leftrightarrow F(\omega)$ and theorem (2-44) it follows that $\overset{*}{f}(t) \leftrightarrow \overset{*}{F}(-\omega)$; therefore the Fourier integral of $|f(t)|^2 = f(t)\overset{*}{f}(t)$ is the function $(1/2\pi)F(\omega) * \overset{*}{F}(-\omega)$; i.e.,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(y)\overset{*}{F}[-(\omega - y)]\,dy = \int_{-\infty}^{\infty} |f(t)|^2\,e^{-j\omega t}\,dt \tag{2-76}$$

Putting $\omega = 0$ in (2-76), we obtain (2-75), because

$$F(y)\overset{*}{F}(y) = A^2(y)$$

A. Ideal low-pass filter. A filter whose amplitude is constant for $|\omega| < \omega_c$ and zero for $|\omega| > \omega_c$ is called ideal low-pass (Fig. 6-4).

$$A(\omega) = \begin{Bmatrix} A_0 & \text{for } |\omega| < \omega_c \\ 0 & \text{for } |\omega| > \omega_c \end{Bmatrix} = A_0 p_{\omega_c}(\omega)$$

Its system function is given by

$$H(\omega) = A_0 p_{\omega_c}(\omega) e^{-j\omega t_0} \tag{6-20}$$

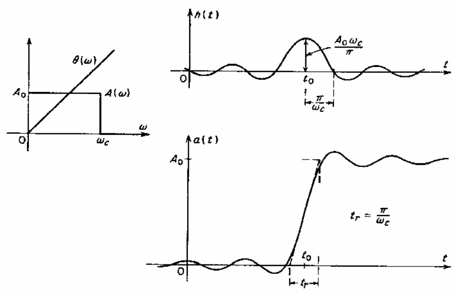


FIGURE 6-4

where $p_{\omega_c}(\omega)$ is a rectangular pulse, and its impulse response by

$$h(t) = \frac{A_0}{\pi}\int_0^{\omega_c} \cos\omega(t - t_0)\,d\omega = \frac{A_0 \sin\omega_c(t - t_0)}{\pi(t - t_0)} \tag{6-21}$$

with $h_{\max} = A_0\omega_c/\pi$ and rise time $t_r = \pi/\omega_c$. To obtain the step response $a(t)$, we use (6-21) and (6-14):

$$a(t) = \frac{A_0}{2} + \frac{A_0}{\pi}\int_0^{t-t_0} \frac{\sin\omega_c\tau}{\tau}\,d\tau = \frac{A_0}{2}\left\{1 + \frac{2}{\pi}Si[\omega_c(t - t_0)]\right\} \tag{6-22}$$

E. Gaussian filter. The filter

$$H(\omega) = A_0 e^{-\alpha\omega^2} e^{-j\omega t_0} \tag{6-45}$$

shown in Fig. 6-15*a* is called Gaussian. To determine its impulse response $h(t)$, we use the result in (2-69)

$$h(t) = \frac{A_0}{\pi}\int_0^{\infty} e^{-\alpha\omega^2}\cos\omega(t - t_0)\,d\omega = \frac{A_0}{2\sqrt{\pi\alpha}}\,e^{-(t-t_0)^2/4\alpha} \tag{6-46}$$

The maximum $h_{\max}$ of $h(t)$ and the rise time t_r [see (6-15)] are given by (Fig. 6-15*b*)

$$h_{\max} = A_0/2\sqrt{\pi\alpha} \qquad t_r = 2\sqrt{\pi\alpha} \tag{6-47}$$

The step response is best obtained from the above and (6-14)

$$a(t) = \frac{A_0}{2} + \frac{A_0}{2\sqrt{\pi\alpha}}\int_0^{t-t_0} e^{-\tau^2/4\alpha}\,d\tau \tag{6-48}$$

and can be expressed in terms of the tabulated *error function erf x* defined by

$$erf\ x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy \tag{6-49}$$

Inserting into (6-48), we obtain the function

$$a(t) = \frac{A_0}{2}\left(1 + erf\,\frac{t - t_0}{2\sqrt{\pi\alpha}}\right) \tag{6-50}$$

shown in Fig. 6-15*c*. As is proved in Sec. 4-4, this filter has the property of minimizing the product of the RMS durations of $h(t)$ and $A(\omega)$.

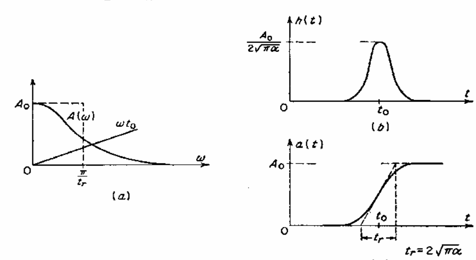
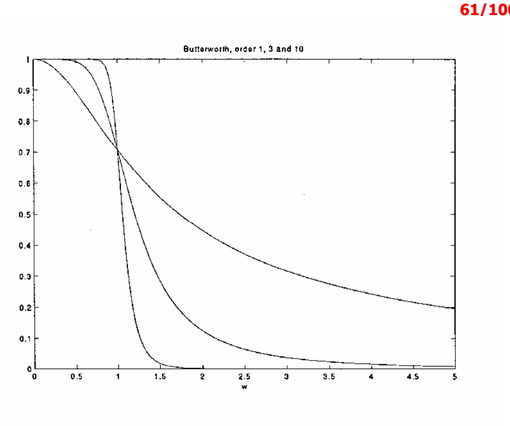
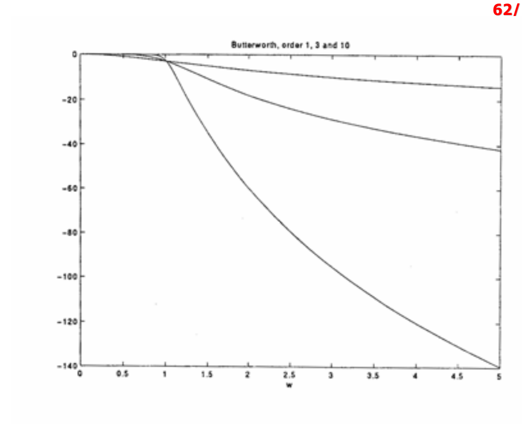
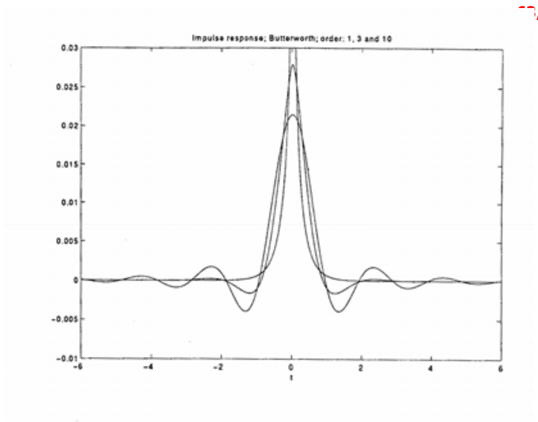
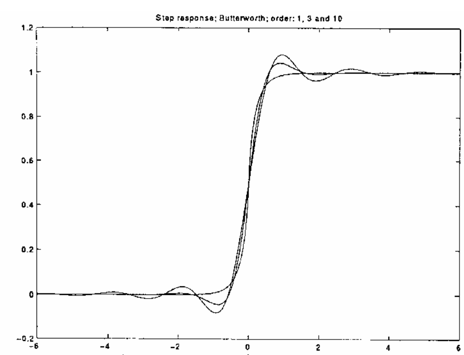


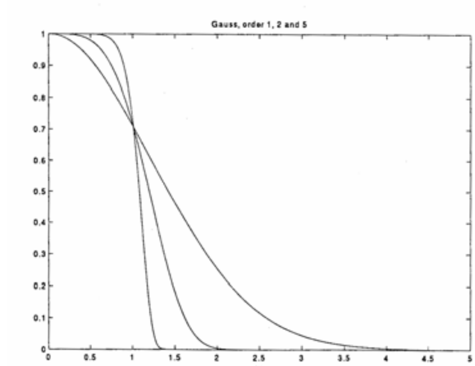
FIGURE 6-15

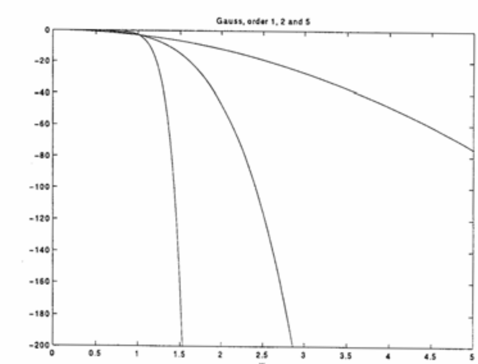

Butterworth, order 1, 3 and 10
w


Butterworth, order 1, 3 and 10
w


Impulse response; Butterworth; order: 1, 3 and 10
t


Step response; Butterworth; order: 1, 3 and 10
t


Gauss, order 1, 2 and 5
w


Gauss, order 1, 2 and 5
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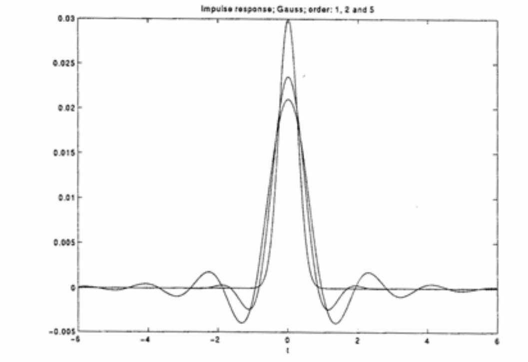
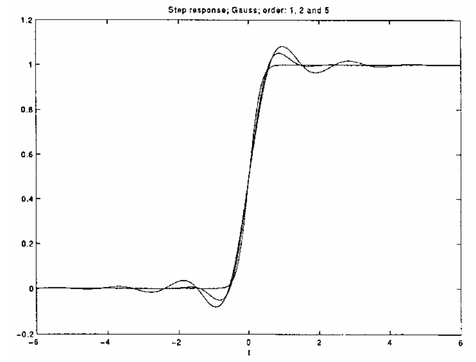




Hilbert transforms. In the following we shall give explicit equations relating the real and imaginary parts of a causal system function. If the causal function $h(t)$ contains no singularities at the origin, then with $H(\omega) = R(\omega) + jX(\omega)$ its Fourier integral, $R(\omega)$ and $X(\omega)$ satisfy the equations

$$X(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(y)}{\omega - y}\, dy \qquad (10\text{-}28)$$

$$R(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{X(y)}{\omega - y}\, dy \qquad (10\text{-}29)$$

known as Hilbert transforms.

First Proof: Convolution Theorem. We denote by $h_e(t)$ and $h_0(t)$ the even and odd parts of $h(t)$ shown in Fig. 10-2; since $h_e(t) = h_0(t)$ for $t > 0$ and $h_e(t) = -h_0(t)$ for $t < 0$, we conclude that

$$h_0(t) = h_e(t) \operatorname{sgn} t \qquad (10\text{-}30)$$

$$h_e(t) = h_0(t) \operatorname{sgn} t \qquad (10\text{-}31)$$

where sgn t is the sign function of Fig. 2-8. The Fourier integrals of $h_e(t)$ and $h_0(t)$ are $R(\omega)$ and $jX(\omega)$ respectively, and the Fourier integral of sgn t equals $2/j\omega$:

$$h_e(t) \to R(\omega) \qquad h_0(t) \to jX(\omega) \qquad \operatorname{sgn} t \to \frac{2}{j\omega} \qquad (10\text{-}32)$$

Since $h_0(t)$ is the product of $h_e(t)$ and sgn t, we conclude from the frequency convolution theorem (2-74) that

$$jX(\omega) = \frac{1}{2\pi} R(\omega) * \frac{2}{j\omega}$$

from which (10-28) follows. We similarly obtain (10-29) from (10-31).

Hilbert transforms. If the function $H(\omega)$ is minimum-phase-shift, then $\ln H_I(p)$ is analytic in the right-hand plane; in this case the attenuation and phase of $H(\omega) = e^{-\alpha(\omega)-j\theta(\omega)}$ are related by the following set of equations similar to (10-28) and (10-29):

$$\theta(\omega_0) = \frac{\omega_0}{\pi}\int_{-\infty}^{\infty} \frac{\alpha(\omega)}{\omega^2 - \omega_0^2}\, d\omega \qquad (10\text{-}67)$$

$$\alpha(\omega_0) = \alpha(0) - \frac{\omega_0^2}{\pi}\int_{-\infty}^{\infty} \frac{\theta(\omega)}{\omega(\omega^2 - \omega_0^2)}\, d\omega \qquad (10\text{-}68)$$

Thus $\theta(\omega)$ can be uniquely determined from $\alpha(\omega)$, and for the determination of $\alpha(\omega)$ one needs not only $\theta(\omega)$ but also the constant $\alpha(0)$. From the proof it will become clear that the above equations are not the only ones relating $\alpha(\omega)$ to $\theta(\omega)$; other sets of similar relationships can be derived (see H. W. Bode, op. cit.).

Example 10-9. Consider a causal low-pass minimum-phase-shift filter with amplitude characteristic

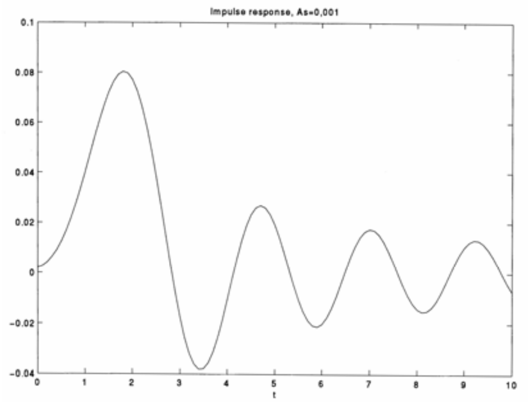
$$A(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ A_s & |\omega| > \omega_c \end{cases}$$

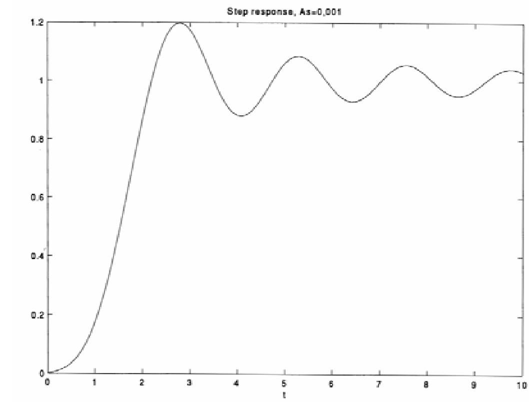
as in Fig. 10-11*a*. Its phase shift $\theta(\omega)$ cannot be assigned arbitrarily but must be given by (10-67)

$$\theta(\omega) = \frac{-2\omega}{\pi}\int_{-\infty}^{\infty} \frac{\ln A_s}{y^2 - \omega_c^2}\, dy = \frac{\ln A_s}{\pi} \ln\left|\frac{\omega - \omega_c}{\omega + \omega_c}\right|$$

as shown in Fig. 10-11*b*. The resulting group delay in the passband, equals

$$t_{gr}(\omega) = \theta'(\omega) = \frac{2 \ln A_s}{\pi}\frac{\omega_c}{\omega^2 - \omega_c^2} \qquad |\omega| < \omega_c$$





Thus the group delay is proportional to the attenuation in the stop band and it tends to infinity as A_s tends to zero. The quantity

$$t_{gr}(0) = \frac{2\,|\ln A_s|}{\pi \omega_c}$$

equals the delay of the center of gravity of the input as it passes through the filter, and it is proportional to $|\ln A_s|$.

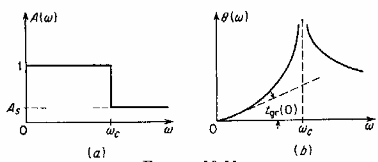


FIGURE 10-11

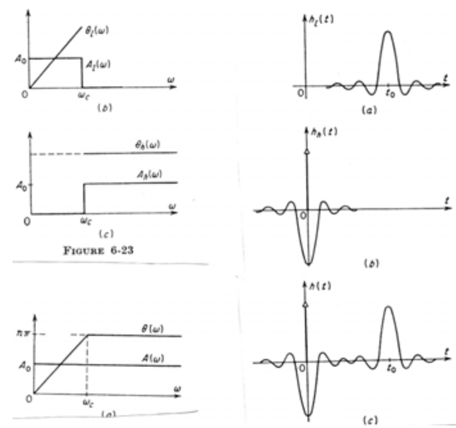


FIGURE 6-23

Example 10-10. As an application of (10-68), we shall evaluate the attenuation of a filter with a phase shift given by the curve

$$\theta(\omega) = \begin{cases} t_0\omega & |\omega| < \omega_c \\ t_0\omega_c & \omega > \omega_c \end{cases}$$

of Fig. 10-12*a*. A noncausal filter with the same phase was discussed in Sec. 6-4. Inserting the above characteristic into (10-68), we obtain

$$\alpha(\omega) = \alpha(0) - \frac{2\omega^2}{\pi}\int_0^{\omega_c} \frac{t_0\,dy}{(y^2 - \omega^2)} - \frac{2\omega^2}{\pi}\int_{\omega_c}^{\infty} \frac{t_0\omega_c\,dy}{y(y^2 - \omega^2)}$$

$$= \alpha(0) + \frac{t_0\omega_c}{\pi}\left[\left(1 + \frac{\omega}{\omega_c}\right)\ln\left(1 + \frac{\omega}{\omega_c}\right) + \left(1 - \frac{\omega}{\omega_c}\right)\ln\left|1 - \frac{\omega}{\omega_c}\right|\right]$$

The corresponding amplitude $A(\omega) = e^{-\alpha(\omega)}$ is shown in Fig. 10-12*b* for $t_0\omega_c = \pi$.

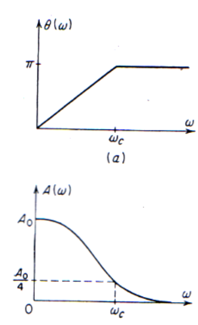


FIGURE 10-12

Specification of $\alpha(\omega)$ and $\theta(\omega)$ in different parts of the ω axis. We shall now show that $H(\omega)$ is uniquely determined if $\alpha(\omega)$ and $\theta(\omega)$ are specified in complementary parts of the ω axis. We shall first consider the case

$$\begin{aligned} &\alpha(\omega) \quad \text{given for } |\omega| < \omega_c \\ &\theta(\omega) \quad \text{given for } |\omega| > \omega_c \end{aligned} \tag{10-81}$$

To find the unknown parts of $\alpha(\omega)$ and $\theta(\omega)$, we shall use a modified form of the Hilbert transforms. We note first that Eqs. (10-67) and (10-68) establish the ω-axis relationship between the real and imaginary parts of a function that is analytic in the right-hand plane. Our problem will therefore be solved if we can find a function whose real part depends on the available information (10-81).

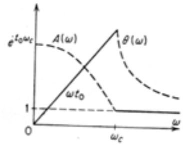


FIGURE 10-13

$A(\omega) = 1$
$\omega > \omega_c$
$\theta(\omega) = \omega t_0$
$\omega < \omega_c$

Example 10-11. We shall use the above result (10-84) to determine a causal, low-pass filter, with linear phase shift $\theta(\omega) = \omega t_0$ in the bandpass $|\omega| < \omega_c$, and a constant amplitude in the stopband $|\omega| > \omega_c$ as in Fig. 10-13. We shall assume, without loss of generality, that $A(\omega) = 1$ for $|\omega| > \omega_c$. With this assumption, the second integral in (10-84) vanishes, and the unknown parts of $\alpha(\omega)$ and $\theta(\omega)$ are given by

$$\frac{-2t_0\sqrt{\omega_c^2 - \omega^2}}{\pi}\int_0^{\omega_c} \frac{y^2\, dy}{\sqrt{\omega_c^2 - y^2}(y^2 - \omega^2)} = \begin{cases} \alpha(\omega) & 0 < \omega < \omega_c \\ \theta(\omega) & \omega > \omega_c \end{cases}$$

The above integral can be easily evaluated; the result is

$$\alpha(\omega) = -t_0\sqrt{\omega_c^2 - \omega^2} \qquad 0 < \omega < \omega_c$$

$$\theta(\omega) = t_0\omega - t_0\sqrt{\omega^2 - \omega_c^2} \qquad \omega > \omega_c$$

The computed parts of the functions $\theta(\omega)$ and

$$A(\omega) = e^{t_0\sqrt{\omega_c^2 - \omega^2}}$$

are shown by the dotted lines in Fig. 10-13.

Ideal High-pass Filter. The frequency characteristics of an ideal high-pass filter are given by

$$A(\omega) = \begin{cases} 0 & \text{for } |\omega| < \omega_c \\ A_0 & \text{for } |\omega| > \omega_c,\ \theta(\omega) = \omega t_0 \end{cases} \tag{6-28}$$

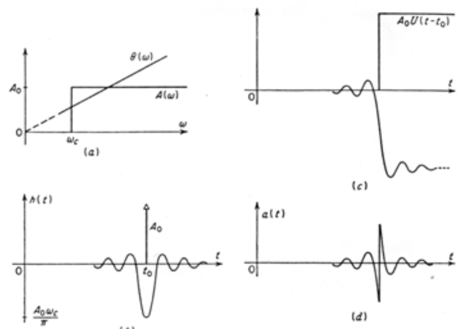


FIGURE 6-7

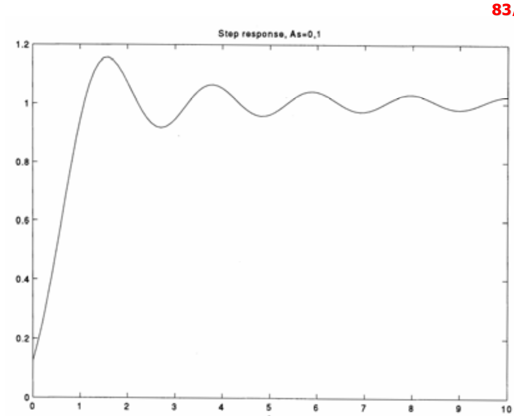
Paley-Wiener condition.† A necessary and sufficient condition for a square-integrable function $A(\omega) \geq 0$ to be the Fourier spectrum of a causal function is the convergence of the integral

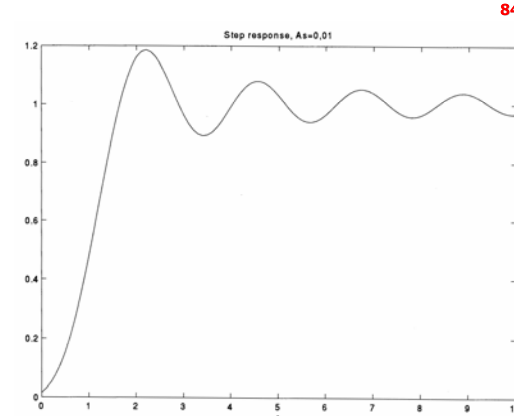
$$\int_{-\infty}^{\infty} \frac{|\ln A(\omega)|}{1 + \omega^2}\, d\omega < \infty \tag{10-90}$$

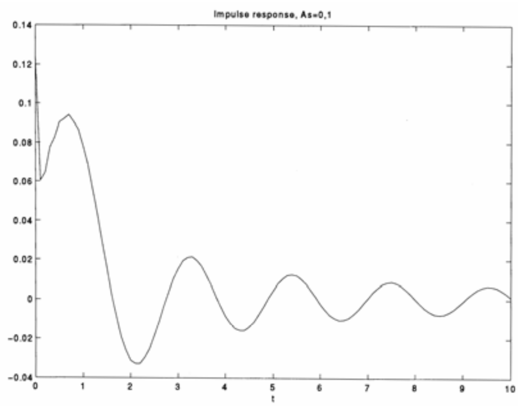
We remark that if the amplitude of a function $H(\omega)$ satisfies (10-90), it does not follow that $H(\omega)$ has a causal inverse. The above says that to $|H(\omega)| = A(\omega)$ a suitable phase can be associated, so that the resulting function has a causal inverse. We further note that if $A(\omega)$ is not square-integrable, then (10-90) is neither necessary nor sufficient.

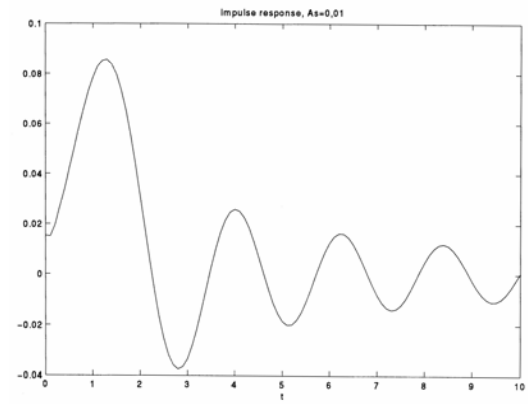
We shall attempt to give a simple justification of (10-90).

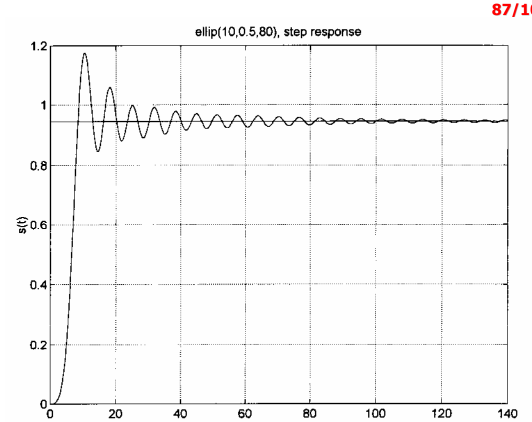
† Raymond E. A. C. Paley and Norbert Wiener, "Fourier Transforms in the Complex Domain," American Mathematical Society Colloquium Publication 19, New York, 1934.

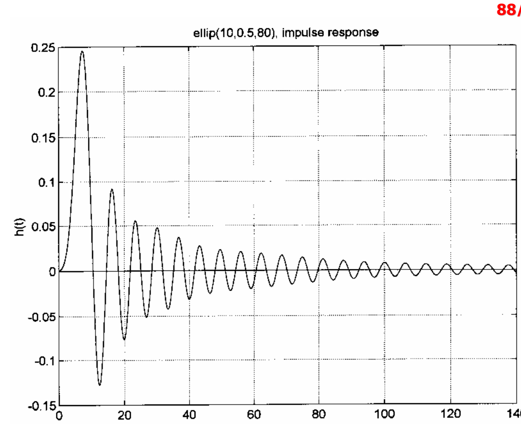


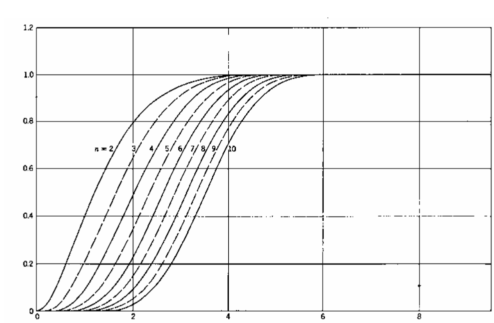




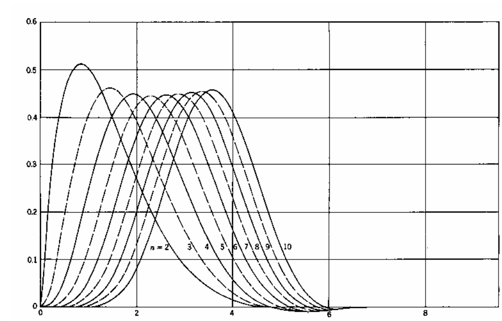






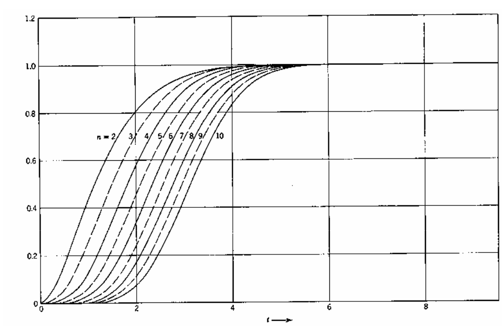


Curve 12. Step response for maximally flat delay (Bessel) filters.



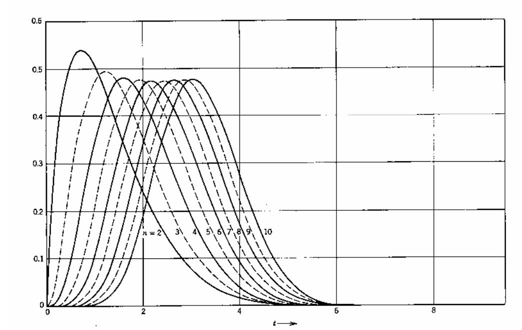
Curve 11. Impulse response for maximally flat delay (Bessel) filters.

91/100



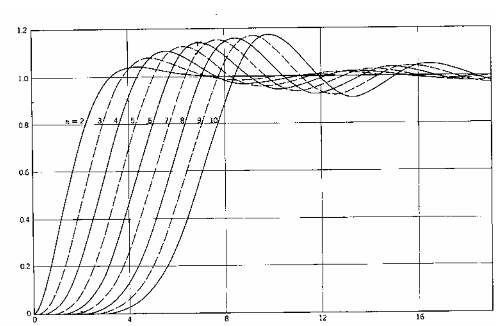
Curve 10. Step response for Gaussian magnitude filters.

92/100



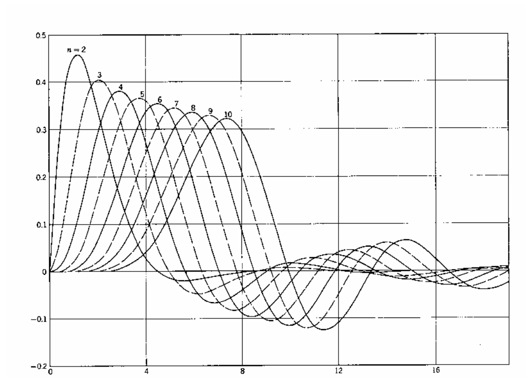
Curve 9. Impulse response for Gaussian magnitude filters.

93/100



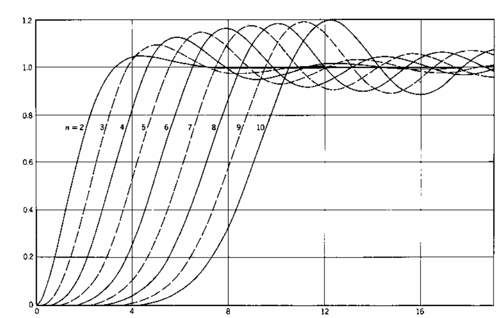
Curve 2. Step response for Butterworth filters.

94/100



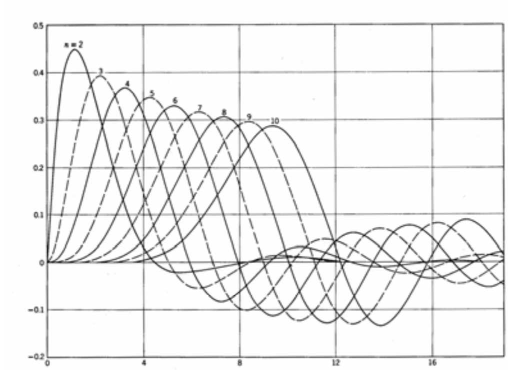
Curve 1. Impulse response for Butterworth filters.

95/100

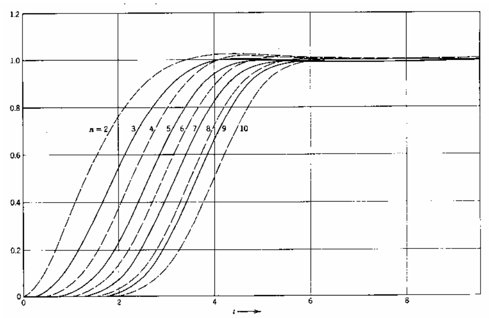


Curve 4. Step response for Chebyshev filters with 0.01 dB ripple.

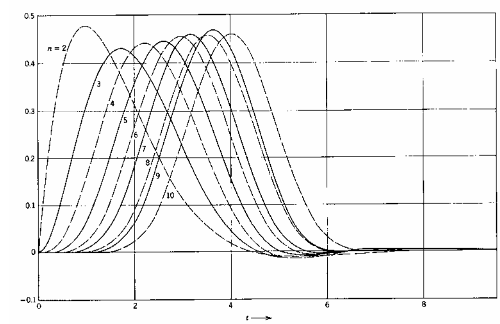
96/100



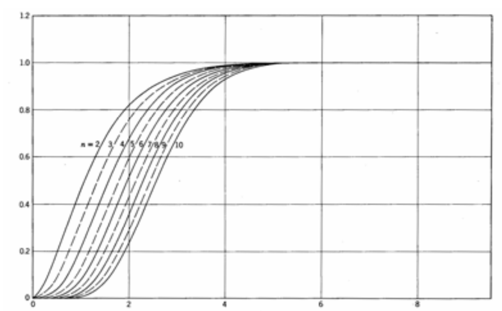
Curve 3. Impulse response for Chebyshev filters with 0.01 dB ripple.



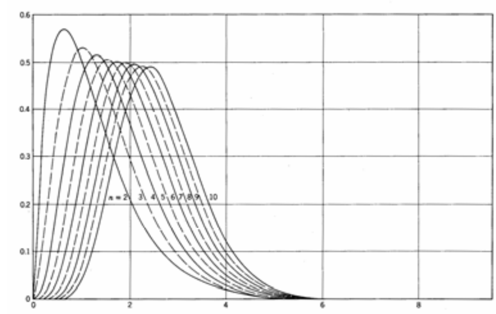
Curve 16. Step response for linear phase with equiripple error filters (phase error = 0.5°).



Curve 15. Impulse response for linear phase with equiripple error filters (phase error = 0.5°).



Curve 24. Step response for synchronously tuned filters.



Curve 23. Impulse response for synchronously tuned filters.